ECE 307 – Techniques for Engineering Decisions

Networks and Flows

George Gross

Department of Electrical and Computer Engineering

University of Illinois at Urbana-Champaign

NETWORKS AND FLOWS

- A network is a system of lines or channels connecting different points
- Examples abound in nearly all aspects of life:

  - electrical systems
  - communication networks
  - airline webs

  - local area networks
  - distribution systems

NETWORKS AND FLOWS

- The network structure is also common to many other systems that at first glance are not necessarily viewed as networks
  - distribution system consisting of manufacturing plants, warehouses and retail outlets
  - matching problems such as work to people, assignments to machines and computer dating

NETWORKS AND FLOWS

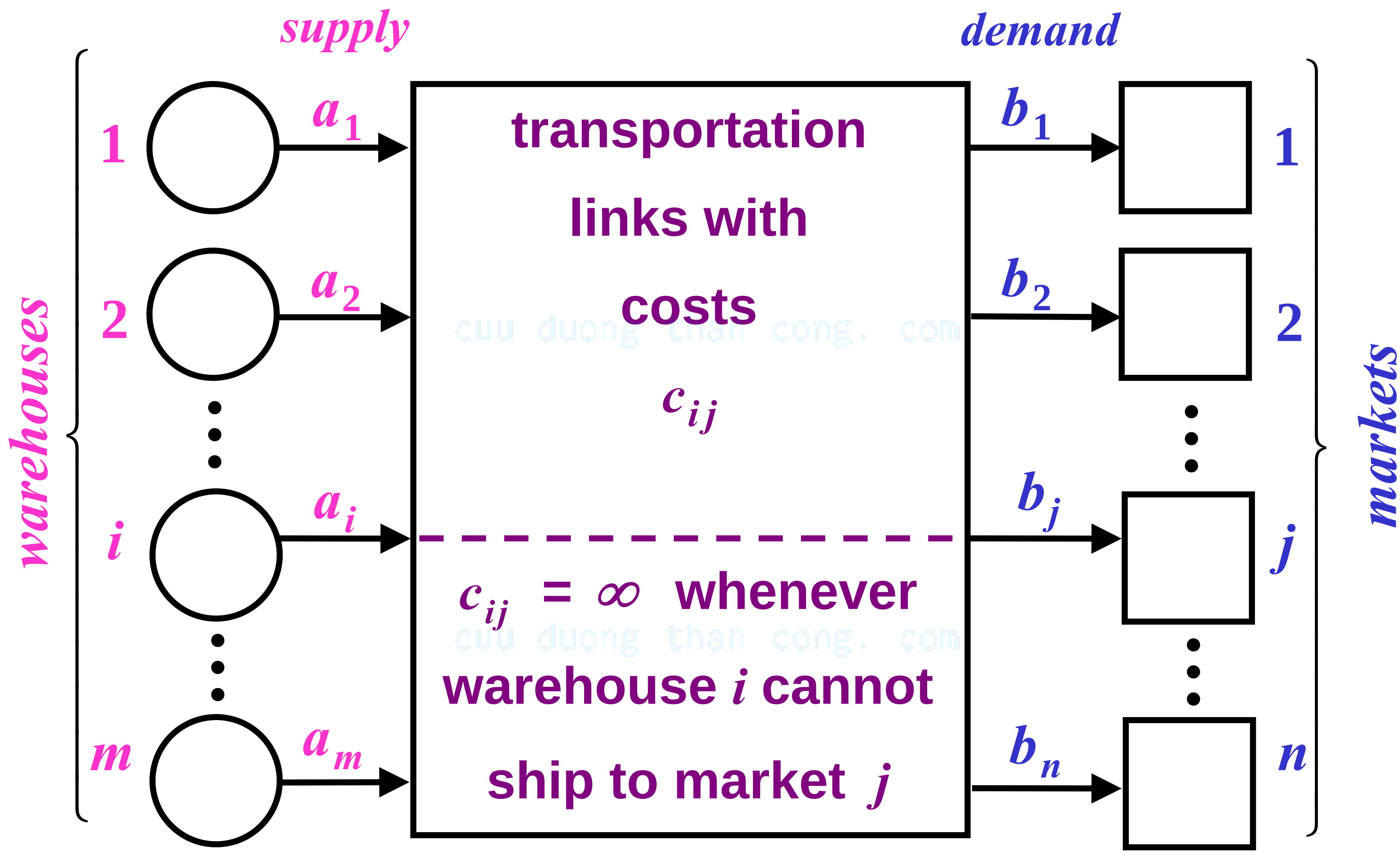
- river systems with pondage for electricity generation
- mail delivery networks
- project management of multiple tasks in a large undertaking such as construction or a space flight

- We consider a broad range of network and network flow problems

THE TRANSPORTATION PROBLEM

- **The basic idea of the transportation problem is illustrated with the problem of distribution of a specified *homogenous* product from several sources to a number of localities *at least cost***
- **We consider a system with m warehouses, n markets and links between them with the specified costs of transportation**

THE TRANSPORTATION PROBLEM



THE TRANSPORTATION PROBLEM

- all the supply comes from the m warehouses; we associate the index $i = 1, 2, \ldots , m$ with a warehouse

- all the demand is at the n markets; we associate the index $j = 1, 2, \ldots , n$ with a market

- shipping costs c_{ij} for each unit from the warehouse i to the market j

THE TRANSPORTATION PROBLEM

- **The transportation problem is to determine the *optimal shipping schedule* that minimizes shipping costs for the set of m warehouses to the set of n markets : the quantities shipped from the warehouse i to each market j**

LP FORMULATION OF THE TRANSPORTATION PROBLEM

- **The decision variables are**

$$x_{ij} = quanity\ shipped\ from\ warehouse\ i\ to\ market\ j$$

$$i = 1, 2, \ldots, m$$

$$j = 1, 2, \ldots, n$$

- **The objective function is**

$$min \quad \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

LP FORMULATION OF THE TRANSPORTATION PROBLEM

- The constraints are:

$$\sum_{j=1}^{n} x_{ij} \leq a_i \qquad i = 1, 2, \ldots, m$$

$$\sum_{i=1}^{m} x_{ij} \geq b_j \qquad j = 1, 2, \ldots, n$$

$$x_{ij} \geq 0 \qquad i = 1, 2, \ldots, m \qquad j = 1, 2, \ldots, n$$

LP FORMULATION OF THE TRANSPORTATION PROBLEM

- **Note that feasibility requires**

$$\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j$$

- **When**

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

every available unit of supply at the m warehouses is shipped to meet all the demands of the n markets; this problem is known as the *standard transportation problem*

STANDARD TRANSPORTATION PROBLEM

$$\min \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \, x_{ij}$$

s.t.

$$\left.\begin{array}{l} \sum_{j=1}^{n} x_{ij} = a_i \\ \sum_{i=1}^{m} x_{ij} = b_j \\ x_{ij} \geq 0 \end{array}\right\} \quad \begin{array}{l} i = 1, \ldots, m \\ j = 1, \ldots, n \end{array}$$

TRANSPORTATION PROBLEM EXAMPLE

w/h i \ *market j*	M_1	M_2	M_3	M_4	*supplies*
W_1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	x_{14} c_{14}	a_1
W_2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	x_{24} c_{24}	a_2
W_3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	x_{34} c_{34}	a_3
demands	b_1	b_2	b_3	b_4	

STANDARD TRANSPORTATION PROBLEM

- The standard transportation problem has
  - mn variables x_{ij}
  - $m + n$ equality constraints

- Since

$$\sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = \sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

there are at most $(m + n - 1)$ independent cons-traints and consequently at most $(m + n - 1)$ independent variables x_{ij}

TRANSPORTATION PROBLEM EXAMPLE

market j / w/h i	M_1	M_2	M_3	M_4	a_i
W_1	2	2	2	1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
b_j	4	3	4	4	

THE LEAST – COST RULE PROCEDURE

- This scheme is used to generate an initial basic feasible solution which has no more than $(m + n - 1)$ positive valued basic variables
- The key idea of the scheme is to select, at each step, the variable x_{ij} with the *lowest shipping costs* c_{ij} as the next basic variable

APPLICATION OF THE LEAST – COST RULE

- c_{14} is the lowest c_{ij} and we select x_{14} as a *basic variable*
- We choose x_{14} as large as possible without violating any constraints:

$$min \{ a_1 , b_4 \} = min \{ 3 , 4 \} = 3$$

- We set $x_{14} = 3$ and

$$x_{11} = x_{12} = x_{13} = 0$$

- We delete row 1 from any further consideration since all the supplies from W_1 are exhausted

APPLICATION OF THE LEAST – COST RULE

market j / *w/h i*	M_1	M_2	M_3	M_4	a_i
W_1	2	2	2	(3) 1	3
W_2	10	8	5	4	7
W_3	7	6	6	8	5
b_j	4	3	4	4	

APPLICATION OF THE LEAST – COST RULE

- The remaining demand at M_4 is

$$4 - 3 = 1$$

which is the value for the modified demand at M_4

- We again apply the *criterion selection* for the reduced tableau: c_{24} is the lowest-valued c_{ij} with $i = 2,\ j = 4$ and we select x_{24} as a *basic variable*

APPLICATION OF THE LEAST – COST RULE

- We choose x_{24} as large as possible without violating any constraints:

$$min \{ a_2 , b_4 \} = min \{ 7 , 1 \} = 1$$

and we set $x_{24} = 1$ and

$$x_{34} = 0$$

- We delete column 4 from any further consi-deration since all the demand at M_4 is exhausted

APPLICATION OF THE LEAST – COST RULE

- **The remaining supply at W_2 is**

$$7 - 1 = 6 \, ,$$

which is the value for the modified supply at W_2

- **We repeat these steps until we find the nonzero**

basic variables **and obtain a** ***basic feasible solution***

- **In the reduced tableau,**

APPLICATION OF THE LEAST – COST RULE

market j / *w/h i*	M_1	M_2	M_3	a_i
W_2	10	8	5 (allocation: 4, circled)	6
W_3	7	6	6 (allocation: *0*)	5
b_j	4	3	4	

APPLICATION OF THE LEAST – COST RULE

- pick x_{23} to enter the basis
- set

$$x_{23} = min\{6, 4\} = 4$$

and set $x_{33} = 0$

- eliminate column 3 and reduce the supply at W_2 to

$$6 - 4 = 2$$

- For the reduced tableau

APPLICATION OF THE LEAST – COST RULE

market j / w/h i	M_1	M_2	a_i
W_2	10	*0* 8	2
W_3	7	3 (circled) 6	5
b_j	4	3	

APPLICATION OF THE LEAST – COST RULE

- **pick x_{32} to enter the basis**
- **set**

$$x_{32} = min \{ 3, 5 \} = 3$$

and set $x_{22} = 0$

- **eliminate column 2 in the reduced tableau**

and reduce the supply at W_3 to

$$5 - 3 = 2$$

- **The last reduced tableau is**

APPLICATION OF THE LEAST – COST RULE

market j / w/h i	M_1	a_i
W_2	10	2
W_3	(2) 7	2
b_j	4	

APPLICATION OF THE LEAST – COST RULE

- **pick x_{31} to enter the basis**
- **set**

$$x_{31} = min \{ 2, 5 \} = 2$$

- **reduce the demand at M_1 to**

$$4 - 2 = 2$$

- **the value of**

$$x_{21} = 2$$

is obtained by default

INITIAL *BASIC FEASIBLE SOLUTION*

w/h i \ market j	M_1	M_2	M_3	M_4	a_i
W_1	(cost 2)	(cost 2)	(cost 2)	3 (cost 1)	3
W_2	2 (cost 10)	(cost 8)	4 (cost 5)	1 (cost 4)	7
W_3	2 (cost 7)	3 (cost 6)	(cost 6)	(cost 8)	5
b_j	4	3	4	4	

APPLICATION OF THE LEAST – COST RULE

- The feasible solution involves only the basic variables and results in shipment costs of

$$\sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij} = 1 \cdot 3 + 4 \cdot 1 + 5 \cdot 4 + 6 \cdot 3 + 7 \cdot 2 + 10 \cdot 2$$

$$= 79$$

THE STANDARD TRANSPORTATION PROBLEM

- The primal problem is

$$\left.\begin{array}{llll} & \min Z = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}\, x_{ij} & & \\ & s.t. & & \\ u_i \leftrightarrow & \sum_{j=1}^{n} x_{ij} = a_i & i = 1, \ldots, m & \\ v_j \leftrightarrow & \sum_{i=1}^{m} x_{ij} = b_j & j = 1, \ldots, n & \\ & x_{ij} \geq 0 & & \end{array}\right\} (P)$$

THE STANDARD TRANSPORTATION PROBLEM

- The dual problem is

$$\left.\begin{aligned} & \max W = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j \\ & s.t. \\ x_{ij} \leftrightarrow \quad & u_i + v_j \le c_{ij} \qquad i = 1, \ldots, m \\ & \qquad\qquad\qquad\quad j = 1, \ldots, n \\ & u_i, v_j \text{ are unrestricted in sign} \end{aligned}\right\} \quad (D)$$

THE STANDARD TRANSPORTATION PROBLEM

- The *complementary slackness conditions* for (D) are

$$x^{*}_{ij}[u^{*}_{i} + v^{*}_{j} - c^{*}_{ij}] = 0 \qquad \begin{matrix} i = 1, \dots, m \\ j = 1, \dots, n \end{matrix}$$

- Due to the equalities in (P), the other *complementary slackness conditions* fail to provide any additional useful information

THE TRANSPORTATION PROBLEM

- The *complementary slackness conditions* obtain

$$x^*_{ij} > 0 \;\Rightarrow\; u^*_i + v^*_j = c_{ij}$$

$$u^*_i + v^*_j < c_{ij} \;\Rightarrow\; x^*_{ij} = 0$$

- We make use of the duality characteristics to

  develop the $u - v$ *method* for solving the *standard*

  transportation problem

THE $u - v$ METHOD

- **The $u - v$ *method* starts with a *basic feasible solution* for the primal problem, obtains the corresponding dual variables (as if the solution were optimal) and uses the duals to determine the *adjacent basic feasible solution*; the process continues until the optimality condition is satisfied**

THE $u - v$ METHOD

- For a ***basic feasible solution***, **we find the dual variable u_i and v_j using the *complementary slackness conditions***

$$u_i + v_j = c_{ij} \qquad\qquad \forall \ basic \ \ x_{ij}$$

with u_i and v_j being unrestricted in sign

THE $u - v$ METHOD

- **We compute using**

$$\tilde{c}_{ij} = c_{ij} - (u_i + v_j) \qquad \forall \; nonbasic \;\; x_{ij}$$

- **the step is the analogue of computing $\underline{\tilde{c}}^T$ in the simplex tableau approach (relative cost improvement vector)**

- **The *complementary-slackness*-based optimality test is performed :**

$$if \quad \tilde{c}_{ij} \geq 0 \;\; \forall \; nonbasic \; x_{ij} \left[x_{ij} = 0 \right] \quad ,$$

then the *basic feasible solution* is *optimal*

THE $u - v$ METHOD

- Otherwise, some nonbasic variable $x_{\bar{p}\bar{q}} \ni$

$$\tilde{c}_{\bar{p}\bar{q}} = c_{\bar{p}\bar{q}} - (u_{\bar{p}} + v_{\bar{q}}) < 0$$

exists and we determine

$$\tilde{c}_{pq} = \underbrace{\min}_{\substack{\bar{p}\bar{q} \ni x_{\bar{p}\bar{q}} \\ \text{is nonbasic}}} \left\{ \tilde{c}_{\bar{p}\bar{q}} \right\}$$

- We, then, select x_{pq} to become a *basic variable* and repeat the process for this new *basic feasible solution*

STANDARD TRANSPORTATION PROBLEM EXAMPLE

- We apply the $u - v$ scheme to the example previously discussed

- The basic step from the dual formulation is to require

$$u_i + v_j = c_{ij} \qquad \forall \ basic \ x_{ij}$$

STANDARD TRANSPORTATION PROBLEM EXAMPLE

- We start with the ***basic feasible solution*** and apply the ***complementary slackness conditions***

$$\begin{aligned}
u_1 + v_4 &= 1 = c_{14} \\
u_2 + v_4 &= 4 = c_{24} \\
u_2 + v_3 &= 5 = c_{23} \\
u_3 + v_2 &= 6 = c_{32} \\
u_3 + v_1 &= 7 = c_{31} \\
u_2 + v_1 &= 10 = c_{21}
\end{aligned}$$

- We have 6 equations in 7 unknowns and so there is an infinite number of solutions

STANDARD TRANSPORTATION PROBLEM EXAMPLE

- Arbitrarily, we set

$$v_4 = 0$$

and solve the equations above to obtain

$$u_1 = 1$$

$$u_2 = 4$$

$$v_3 = 1$$

$$v_1 = 6$$

$$u_3 = 1$$

$$v_2 = 5$$

STANDARD TRANSPORTATION PROBLEM EXAMPLE

- The $\tilde{c}_{ij}$ for the ***nonbasic variables*** are

$$x_{11}: \quad \tilde{c}_{11} = c_{11} - (u_1 + v_1) = 2 - (1+6) = -5$$

$$x_{12}: \quad \tilde{c}_{12} = c_{12} - (u_1 + v_2) = 2 - (1+5) = -4$$

$$x_{13}: \quad \tilde{c}_{13} = c_{13} - (u_1 + v_3) = 2 - (1+1) = 0$$

$$x_{34}: \quad \tilde{c}_{34} = c_{34} - (u_3 + v_4) = 8 - (1+0) = 7$$

$$x_{33}: \quad \tilde{c}_{33} = c_{33} - (u_3 + v_3) = 6 - (1+1) = 4$$

STANDARD TRANSPORTATION PROBLEM EXAMPLE

- We determine

$$\tilde{c}_{pq} = \underbrace{\min}_{\substack{\bar{p}\bar{q} \ni x_{\bar{p}\bar{q}} \\ is\ nonbasic}} = \tilde{c}_{11} = -5$$

and consequently the *nonbasic variable* x_{11} is

introduced into the *basis*

- We determine the maximal value of x_{11} by

setting $x_{11} = \theta$ and make use of the tableau

STANDARD TRANSPORTATION EXAMPLE

market j / w/h i	M_1	M_2	M_3	M_4	a_i
W_1	θ			$3 - \theta$	3
W_2	$2 - \theta$		4	$1 + \theta$	7
W_3	2	3			5
b_j	4	3	4	4	

STANDARD TRANSPORTATION EXAMPLE

- Therefore,

$$max\ \theta = min\ \{ 2, 3 \} = 2$$

- Consequently, $x_{21} = 0$ and leaves the basis

- We obtain the *basic feasible solution*

$$x_{14} = 1,\ x_{11} = 2,\ x_{31} = 2,\ x_{32} = 3,\ x_{23} = 4,\ x_{24} = 3$$

and repeat to solve the $u - v$ problem for this new *basic feasible solution*

STANDARD TRANSPORTATION EXAMPLE

market j / w/h i	$v_1 = 2$	$v_2 = 1$	$v_3 = 2$	$v_4 = 1$	a_i
$u_1 = 0$	(2) cost 2	cost 2	cost 2	(1) cost 1	3
$u_2 = 3$	cost 10	cost 8	(4) cost 5	(3) cost 4	7
$u_3 = 5$	(2) cost 7	(3) cost 6	cost 6	cost 8	5
b_j	4	3	4	4	

STANDARD TRANSPORTATION EXAMPLE

- The complementary slackness conditions of the nonzero valued basic variables obtain

$$u_1 + v_1 = c_{11} = 2$$

$$u_1 + v_4 = c_{14} = 1$$

$$u_2 + v_3 = c_{23} = 5$$

$$u_2 + v_4 = c_{24} = 4$$

$$u_3 + v_1 = c_{31} = 7$$

$$u_3 + v_2 = c_{32} = 6$$

STANDARD TRANSPORTATION EXAMPLE

- We set

$$u_1 = 0$$

and therefore

$$v_3 = 2 \qquad v_1 = 2$$

$$u_3 = 5 \qquad u_3 = 5$$

$$v_2 = 1 \qquad v_2 = 0$$

- We compute $\tilde{c}_{ij}$ for each nonbasic variable x_{ij}

STANDARD TRANSPORTATION EXAMPLE

$$\tilde{c}_{12} = c_{12} - (u_1 + v_2) = 2 - (0+1) = 1$$

$$\tilde{c}_{13} = c_{13} - (u_1 + v_3) = 2 - (0+2) = 0$$

$$\tilde{c}_{21} = c_{21} - (u_2 + v_1) = 10 - (3+2) = 5$$

$$\tilde{c}_{22} = c_{22} - (u_2 + v_2) = 8 - (3+1) = 4$$

$$\tilde{c}_{33} = c_{33} - (u_3 + v_3) = 6 - (5+2) = -1$$

$$\tilde{c}_{34} = c_{34} - (u_3 + v_4) = 8 - (5+1) = 2$$

only possible improvement (→ -1)

- We introduce x_{33} as a *basic variable* and determine its *nonnegative value* θ in the tableau

STANDARD TRANSPORTATION EXAMPLE

market j / w/h i	M_1	M_2	M_3	M_4	a_i
W_1	$2 + \theta$			$1 - \theta$	3
W_2			$4 - \theta$	$3 + \theta$	7
W_3	$2 - \theta$	3	θ		5
b_j	4	3	4	4	

STANDARD TRANSPORTATION EXAMPLE

- The limiting value of θ is

$$\theta = min\{2, 4, 1\} = 1$$

- Consequently, x_{14} leaves the basis and x_{33}

  enters the basis with the value 1

- We obtain the adjacent basic feasible solution in

STANDARD TRANSPORTATION EXAMPLE

w/h i \ market j	$v_1 = 2$	$v_2 = 1$	$v_3 = 1$	$v_4 = 0$	a_i
$u_1 = 0$	(3) 2	2	2	1	3
$u_2 = 4$	10	8	(3) 5	(4) 4	7
$u_3 = 5$	(1) 7	(3) 6	(1) 6	8	5
b_j	4	3	4	4	

STANDARD TRANSPORTATION EXAMPLE

- We evaluate $\tilde{c}_{ij}$ for each nonbasic variable;

  $\tilde{c}_{ij} \geq 0$ and so we have an optimal solution with

  shipping 3 from W_1 to M_1 with costs 6

  shipping 1 from W_3 to M_1 with costs 7

  shipping 3 from W_3 to M_2 with costs 18

  shipping 1 from W_3 to M_3 with costs 6

  shipping 3 from W_2 to M_3 with costs 15

  shipping 4 from W_2 to M_4 with costs 16

  and resulting in the least total costs of 68

ELECTRICITY DISTRIBUTION EXAMPLE

- We consider in an electric utility system in which

  3 power plants are used to supply the demand of

  4 cities

- The supplies available from the 3 plants are given
- The demands of the 4 cities are specified
- The costs of supplying each $10^6\ kWh$ are given

ELECTRICITY COSTS

from \ *to*		city 1	city 2	city 3	city 4	*supplies* (10^6 *kWh*)
plant	1	8	6	10	9	35
	2	9	12	13	7	50
	3	14	9	16	5	40
demands (10^6 *kWh*)		45	20	30	30	125

ELECTRICITY COSTS

from \ *to*	*city* 1	2	3	4	*supplies* (10^6 *kWh*)
pl				9	35
				7	50
				5	40
demands (10^6 *kWh*)	45	20	30	30	125

balanced transportation problem

ELECTRICITY ALLOCATION EXAMPLE

- We note that

$$\sum_{i=1}^{3} a_i = \sum_{j=1}^{4} b_j$$

and so we have a balanced transportation problem

- We find a basic feasible solution using the least-cost rule

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

from \ *to*		*city* 1	2	3	4	*supplies* (10^6 *kWh*)
plant	1	8	6	10	*0* 9	35
	2	9	12	13	*0* 7	50
	3	14	9	16	30 5	10
demands (10^6 *kWh*)		45	20	30	30	125

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

- And we set

$$x_{34} = 30$$

$$x_{14} = 0$$

$$x_{24} = 0$$

- We compute the supply left at plant 3 and remove

  column 4 from further consideration

- We continue with the reduced system

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

from \ to		city 1	city 2	city 3	supplies (10^6 kWh)
plant	1	8	20 / 6	10	15
plant	2	9	*0* / 12	13	50
plant	3	14	*0* / 9	16	10
demands (10^6 kWh)		45	20	30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and so we set

$$x_{12} = 20$$

$$x_{22} = 0$$

$$x_{32} = 0$$

- We recompute the supply left at plant 1 and

  remove column 2 from further consideration

- The new reduced system obtains

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

to / *from*		*city* 1	*city* 3	*supplies* (10^6 *kWh*)
plant	1	15 [8]	*0* [10]	15
	2	[9]	[13]	50
	3	[14]	[16]	10
demands (10^6 *kWh*)		30	30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and therefore we set

$$x_{11} = 15$$

$$x_{13} = 0$$

and remove row 1 from further consideration

since the supply at plant 1 is exhausted

- **The operation is repeated for the further reduced system**

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

from \ *to*		*city* 1	*city* 3	*supplies* (10^6 *kWh*)
plant	2	30 [9]	[13]	20
	3	*0* [14]	[16]	10
demands (10^6 *kWh*)		30	30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

and therefore we set

$$x_{21} = 30$$

$$x_{31} = 0$$

and remove column 1 from further consideration

since all the demand in city 1 is satisfied

- We are finally left with

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

from \ to		city 3	supplies (10^6 *kWh*)
plant	2	20 (13)	20
	3	10 (16)	10
demands (10^6 *kWh*)		30	

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

which allows us to set

$$x_{23} = 20$$

$$x_{33} = 10$$

- The basic feasible solution has the costs

$$Z = 30 \cdot 5 + 20 \cdot 6 + 15 \cdot 8 + 30 \cdot 9 + 20 \cdot 13 + 10 \cdot 16 = 1080$$

- We improve this solution by using the $u - v$ scheme
- The first tableau corresponding to the initial basic feasible solution is:

ELECTRICITY ALLOCATION EXAMPLE: SOLUTION

from \ *to*		*city* 1	*city* 2	*city* 3	*city* 4	*supplies* (10^6 *kWh*)
plant	1	15 (8)	20 (6)			35
plant	2	30 (9)		20 (13)		50
plant	3			10 (16)	30 (5)	40
demands (10^6 *kWh*)		45	20	30	30	

STANDARD TRANSPORTATION EXAMPLE

- We compute, the possible improvements at each nonbasic variable:

$$\tilde{c}_{31} = c_{31} - (u_3 + v_1) = 14 - (4+8) = 2$$

$$\tilde{c}_{22} = c_{22} - (u_2 + v_2) = 12 - (1+6) = 5$$

$$\tilde{c}_{32} = c_{32} - (u_3 + v_2) = 9 - (4+6) = -1$$

$$\tilde{c}_{13} = c_{13} - (u_1 + v_3) = 10 - (0+12) = -2$$

$$\tilde{c}_{14} = c_{14} - (u_1 + v_4) = 9 - (0+1) = 8$$

$$\tilde{c}_{24} = c_{24} - (u_2 + v_4) = 7 - (1+1) = 5$$

cuu duong than cong. com

improvement possible ← (−1)

better possibility ← (−2)

STANDARD TRANSPORTATION EXAMPLE

- **We bring x_{13} into the basis and determine the value of θ using the tableau structure**

- **From the tableau we conclude that**

$$\theta = min \{ 15, 20 \} = 15$$

and therefore x_{11} leaves the basis and obtain the adjacent basic possible solution given in the table

STANDARD TRANSPORTATION EXAMPLE

plants \ *cities*	1	2	3	4	a_i
1	$15 - \theta$	20	θ		35
2	$30 + \theta$		$20 - \theta$		50
3			10	30	40
b_j	45	20	30	30	

STANDARD TRANSPORTATION EXAMPLE

- **The adjacent basic feasible solution is**

$$x_{21} = 45, \quad x_{12} = 20, \quad x_{13} = 15, \quad x_{23} = 5, \quad x_{33} = 10, \quad x_{34} = 30$$

and the new value of Z **is**

$$Z = 20 \cdot 6 + 15 \cdot 10 + 45 \cdot 9 + 5 \cdot 13 + 10 \cdot 16 + 30 \cdot 5$$

$$= 1050 < 1080$$

- **We again pursue a** $u - v$ **improvement strategy by starting with the tableau**

STANDARD TRANSPORTATION EXAMPLE

cities / *plants*	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = -1$	*supplies*
$u_1 = 0$		(20) 6	(15) 10		35
$u_2 = 3$	(45) 9		(5) 13		50
$u_3 = 6$			(10) 16	(30) 5	40
demands	45	20	30	30	

STANDARD TRANSPORTATION EXAMPLE

- The complementary slackness conditions obtain the possible improvements

$$
\begin{aligned}
\tilde{c}_{11} &= c_{11} - (u_1 + v_1) = 8 - (0 + 6) = 2 \\
\tilde{c}_{31} &= c_{31} - (u_3 + v_1) = 14 - (6 + 6) = 2 \\
\tilde{c}_{22} &= c_{22} - (u_2 + v_2) = 12 - (3 + 6) = 3 \\
\tilde{c}_{32} &= c_{32} - (u_3 + v_2) = 9 - (6 + 6) = -3 \\
\tilde{c}_{14} &= c_{14} - (u_1 + v_4) = 9 - (0 - 1) = 10 \\
\tilde{c}_{24} &= c_{24} - (u_2 + v_4) = 7 - (3 - 1) = 5
\end{aligned}
$$

only possible improvement (← -3)

- We bring x_{32} into the basis and determine its value θ using

STANDARD TRANSPORTATION EXAMPLE

cities \ plants	1	2	3	4	a_i
1		$20 - \theta$	$15 + \theta$		35
2	45		5		50
3		θ	$10 - \theta$	30	40
b_j	45	20	30	30	

STANDARD TRANSPORTATION EXAMPLE

and so

$$\theta = min\{ 10, 20 \} = 10$$

- **The adjacent basic feasible solution is, then,**

$$x_{21} = 45 \qquad x_{12} = 10 \qquad x_{32} = 10$$

$$x_{13} = 25 \qquad x_{23} = 5 \qquad x_{34} = 30$$

and the value of Z becomes

$$Z = 45 \cdot 9 + 10 \cdot 6 + 10 \cdot 9 + 25 \cdot 10 + 5 \cdot 13 \; 30 \cdot 5 = 1,020$$

- **You are asked to prove, using complementary slackness conditions, that this is the optimum**

NONSTANDARD TRANSPORTATION PROBLEM

- The nonstandard transportation problem arises when supply and demand are unbalanced: either supply exceeds demand or vice versa

- We solve by transforming the nonstandard problem into a standard one

- The approach is to create a fictitious entity and thereby restore the problem to balanced status

NONSTANDARD TRANSPORTATION PROBLEM

- For the case

$$\underbrace{\sum_{i=1}^{m} a_i}_{\text{supply}} > \underbrace{\sum_{j=1}^{n} b_j}_{\text{demand}}$$

we create the fictitious market M_{n+1} to absorb all the excess supply $\left(\sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j \right)$; we set $c_{i,\,n+1} = 0$, $\forall \; i = 1, 2, \ldots, m$ since M_{n+1} does not exist in reality

The problem is then in standard form with $j = 1, \ldots, n+1$, an augmented number of markets

NONSTANDARD TRANSPORTATION PROBLEM

- **For the case**

$$\underbrace{\sum_{j=1}^{n} b_j}_{\text{demand}} > \underbrace{\sum_{i=1}^{m} a_i}_{\text{supply}}$$

the problem is *not* , in effect, *feasible* since all the demands cannot be met and therefore the least-cost shipping schedule is that which will supply as much as possible of the demands of the markets

NONSTANDARD TRANSPORTATION PROBLEM

- **For the overdemand case, we introduce the fictitious warehouse W_{m+1} to supply the shortage; $\left[\sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i \right]$ we set $c_{m+1,j} = 0$ for $j = 1, 2, \ldots, n$ and the problem is in standard form with $i = 1, \ldots, m+1$ (augmented number of warehouses)**

NONSTANDARD TRANSPORTATION PROBLEM

- **Note that the variable $x_{m+1,j}$ is the *shortage* at market j and is the shortfall in the demand b_j experienced by the market M_j due to inadequate supplies**

- **For each market j, $x_{m+1,j}$ is a measure of the *infeasibility* of the problem**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- This problem is concerned with the schedule of 2 plants A and B in the purchase of the raw supplies from 3 growers

grower	*availability* (*ton*)	*price* (*$ / ton*)
Smith	200	10
Jones	300	9
Richard	400	8

EXAMPLE: CANNING OPERATIONS SCHEDULING

and shipping costs in *$/ton* given by

from \ *to*	*plant*	
	A	***B***
Smith	**2**	**2.5**
Jones	**1**	**1.5**
Richard	**5**	**3**

EXAMPLE: CANNING OPERATIONS SCHEDULING

- The plants' labor costs and capacity limits are

plant	*A*	*B*
capacity (*ton*)	450	550
labor costs (*$ / ton*)	25	20

EXAMPLE: CANNING OPERATIONS SCHEDULING

- The selling price for canned goods is $50 \; \$ / ton$ and the company can sell all it produces
- The problem is to determine the *maximum* profit schedule
- Note that this is an unbalanced problem since

$$supply = 200 + 300 + 400 = 900 \; tons$$

$$demand = 450 + 550 = 1000 \; tons > 900 \; tons$$

- Clearly, the decision variables are the amounts purchased from each grower and shipped to each plant

EXAMPLE: CANNING OPERATIONS SCHEDULING

- The objectives is formulated as

$$max\ Z = \left[\underbrace{50-25-10-2}_{13}\right]x_{SA} + \left[\underbrace{50-25-9-1}_{15}\right]x_{JA}$$

$$+ \left[\underbrace{50-25-8-5}_{12}\right]x_{RA} + \left[\underbrace{50-20-10-2.5}_{17.5}\right]x_{SB}$$

$$+ \left[\underbrace{50-20-9-1.5}_{19.5}\right]x_{JB} + \left[\underbrace{50-20-8-3}_{19}\right]x_{RB}$$

EXAMPLE: CANNING OPERATIONS SCHEDULING

- The supply constraints are

$$x_{SA} + x_{SB} \leq 200$$

$$x_{JA} + x_{JB} \leq 300$$

$$x_{RA} + x_{RB} \leq 400$$

- The demand constraints are

$$x_{SA} + x_{JA} + x_{RA} \leq 450$$

$$x_{SB} + x_{JA} + x_{RB} \leq 550$$

EXAMPLE: CANNING OPERATIONS SCHEDULING

- Clearly, all decision variables are nonnegative
- The unbalanced nature of the problem requires the introduction of a *fictitious* grower F with the supply 100 corresponding to the supply shortage; in this way the *nonstandard* problem becomes *standard*
- We set up the standard transportation problem

EXAMPLE: CANNING OPERATIONS SCHEDULING

plant j / *grower i*	*A*	*B*	*supply*
S	13	17.5	200
J	15	19.5	300
R	12	19	400
F	*0*	*0*	100
demand	450	550	

EXAMPLE: CANNING OPERATIONS SCHEDULING

- Please note that the objective is a ***maximization*** rather than a ***minimization***
- We therefore recast the mechanics of the ***u-v*** scheme for the ***maximization*** problem
- As a homework exercise, show that the duality complementary slackness conditions allow us to change the $u - v$ algorithm in the following way:

EXAMPLE: CANNING OPERATIONS SCHEDULING

- the selection of the nonbasic variable x_{ij} to enter the basis is from those x_{ij} where the corresponding

$$c_{ij} > u_i + v_j$$

and we evaluate and focus on all $\tilde{c}_{ij} > 0$ so that x_{ij} is a candidate to enter the basis

- we pick x_{pq}

$$\tilde{c}_{pq} = \underbrace{max}_{\substack{\bar{p}\bar{q} \ni x_{\bar{p}\bar{q}} \\ is\ nonbasic}} \left\{ \tilde{c}_{\bar{p}\bar{q}} \right\}$$

EXAMPLE SOLUTION

plant j / *grower i*	*A*	*B*	*supply*
S	200 (13)	*0*	200
J	250 (15)	50 (19.5)	300
R	*0*	400 (19)	400
F	*0* (*0*)	100 (*0*)	100
demand	450	550	

EXAMPLE SOLUTION

- We construct the $u - v$ relations for this solution

$$u_1 + v_1 = 13 \qquad u_2 + v_2 = 19.5$$

$$u_2 + v_1 = 15 \qquad u_3 + v_2 = 19$$

$$u_4 + v_2 = 0$$

- We arbitrarily set $u_1 = 0$ and compute

$$v_1 = 13,\ u_2 = 2,\ v_2 = 17.5,\ u_3 = 1.5,\ u_4 = -17.5$$

EXAMPLE SOLUTION

- We evaluate the $\tilde{c}_{ij}$ corresponding to the nonbasic variables

$$\tilde{c}_{31} = c_{31} - (u_3 + v_1) = 12 - (1.5 + 13) = -2.5$$

$$\tilde{c}_{41} = c_{41} - (u_4 + v_1) = 0 - (-17.5 + 13) = 4.5$$

$$\tilde{c}_{12} = c_{12} - (u_1 + v_2) = 17.5 - (0 + 17.5) = 0$$

single possible improvement

- Thus, x_{41} enters the basis and we determine θ

EXAMPLE SOLUTION

plant j / *grower i*	*A*	*B*	*supply*
S	200 (13)		200
J	250 – θ (15)	50 + θ (19.5)	300
R		400 (19)	400
F	θ (*0*)	100 – θ (*0*)	100
demand	450	550	

EXAMPLE SOLUTION

- It follows that

$$\theta = min \{ 250, 100 \} = 100$$

and so the adjacent basic feasible solution is

$$x_{11} = 200, \; x_{21} = 150, \; x_{41} = 100, \; x_{22} = 150, \; x_{32} = 400$$

- We repeat the $u - v$ procedure to obtain

EXAMPLE SOLUTION

$$u_1 + v_1 = 13 \qquad u_2 + v_2 = 19.5$$

$$u_2 + v_1 = 15 \qquad u_3 + v_2 = 19$$

$$u_4 + v_1 = 0$$

- We solve by arbitrarily setting $u_1 = 0$ and obtain

$$v_1 = 13 \,,\ u_2 = 2 \,,\ v_2 = 17.5 \,,\ u_3 = 1.5 \,,\ u_4 = -13$$

EXAMPLE SOLUTION

- We compute the $\tilde{c}_{ij}$ for the nonbasic variables

$$\tilde{c}_{12} = 17.5 - (0 + 17.5) = 0$$

$$\tilde{c}_{31} = 12 - (1.5 + 13) = -2.5$$

$$\tilde{c}_{42} = 0 - (-13 + 17.5) = -4.5$$

EXAMPLE SOLUTION

- **Since each $\tilde{c}_{ij}$ is ≤ 0 no more improvement in the maximization is possible and so the maximum profits are**

$$Z = (200)13 + (150)15 + (100)0 + (150)19.5 + (400)19$$

$$= 15,375 \ \$$$

SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

- **The problem is concerned with the weekly production scheduling over a 4 – week period**
  - **production costs from each item**

first two weeks	$ 10
last two weeks	$ 15

  - **demands that need to be met are**

week	1	2	3	4
demand	300	700	900	800

SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

- **weekly plant capacity is 700**
- **overtime is possible for weeks 2 and 3 with**
  - **the production of additional 200 *units***
  - **additional cost per unit of *$* 5**
- ***$* 3 for weekly storage of excess production**
- **the objective is to *minimize* the *total costs* for the 4-week schedule**

- **The decision variables are**

x_{ij} = production in week i for use in week j market

SCHEDULING PROBLEM AS A STANDARD TRANSPORTATION PROBLEM

production \ demand		1	2	3	4	*F*	*supply*
1		10	13	16	19	*0*	700
2	*normal*	*M*	10	13	16	*0*	700
	o/t	*M*	15	18	[illegible]	*0*	200
3	*normal*	*M*	*M*	15	18	*0*	700
	o/t	*M*	[illegible]	[illegible]	[illegible]	*0*	200
4		*M*	*M*	*M*	15	*0*	700
demand		300	700	900	800	500	

M is a very large number

3,200

2,700

3,200 – 2,700

ASSIGNMENT PROBLEM

- We are given

  n machines $M_1, M_2, \ldots, M_n \leftrightarrow i$

  n jobs $J_1, J_2, \ldots, J_n \leftrightarrow j$

  $c_{ij} =$ cost of doing job j on machine i

  $c_{ij} = \hat{M}$ if job j cannot be done on machine i

  each machine can only do one job and we wish to determine the optimal match, i.e., the assignment with the lowest total costs of doing all the jobs j on the n machines available

ASSIGNMENT PROBLEM

- **The brute force approach is simply enumeration: consider $n = 10$ and there are $3{,}628{,}800$ possible choices!**

- **We can, however, introduce *categorical* decision variables**

$$x_{ij} = \begin{cases} 1 & \text{job } j \text{ is assigned to machine } i \\ 0 & \text{otherwise} \end{cases}$$

ASSIGNMENT PROBLEM

and the problem constraints can be stated as

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \quad \text{each machine does exactly 1 job}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \quad \text{each job is assigned}$$

to only 1 machine

- The objective, then, is

$$min\ Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

ASSIGNMENT PROBLEM

- This assignment problem is a standard transportation problem with

$$a_i = 1 \qquad \forall\ i$$

$$b_j = 1 \qquad \forall\ j$$

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j$$

NONSTANDARD ASSIGNMENT PROBLEM

- Suppose we have m machines and n jobs with $m \neq n$
- We may convert this into an equivalent *standard assignment problem* with *equal* number of machines and jobs
- The conversion requires the introduction of either *fictitious* jobs or *fictitious* machines

NONSTANDARD ASSIGNMENT PROBLEM

- **In the case $m > n$:**

  we create ($m - n$) fictitious jobs and we have m machines and $n + m - n = m$ jobs; we assign the machinery costs for the fictitious goods to be 0 : note that there is no change in the objective function since a fictitious job assigned to a machine is, in effect, a machine that is *idle*

NONSTANDARD ASSIGNMENT PROBLEM

- For the case $n > m$:

  we create $(n - m)$ *fictitious* machines with machine costs of 0 and the solution obtained has the $(n - m)$ jobs that cannot be done due to lack of machines

NONSTANDARD ASSIGNMENT PROBLEM

- **In principle, any assignment problem may be solved using the transportation problem technique; in practice , this is not good since there exists *degeneracy* in every basic feasible solution**

- **We note that in the *standard assignment problem* for m machines with $m = n$, there are exactly m x_{ij} that are 1 (nonzero) but every basic feasible solution of the transportation problem has $(2m - 1)$ basic variables and therefore contains $(m - 1)$ zero valued basic variables**