# ECE 307 – Techniques for Engineering Decisions

## Transshipment and Shortest Path Problems

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# TRANSSHIPMENT PROBLEMS

- **We consider the shipment of a** ***homogeneous*** **commodity from a specified point or** ***source*** **to a particular destination or** ***sink***

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- **In general, the source and the sink need not be directly connected; rather, the flow goes through the** ***transshipment points*** **or the intermediate nodes**

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- **The objective is to determine the** ***maximal flow*** **from the source to the sink**

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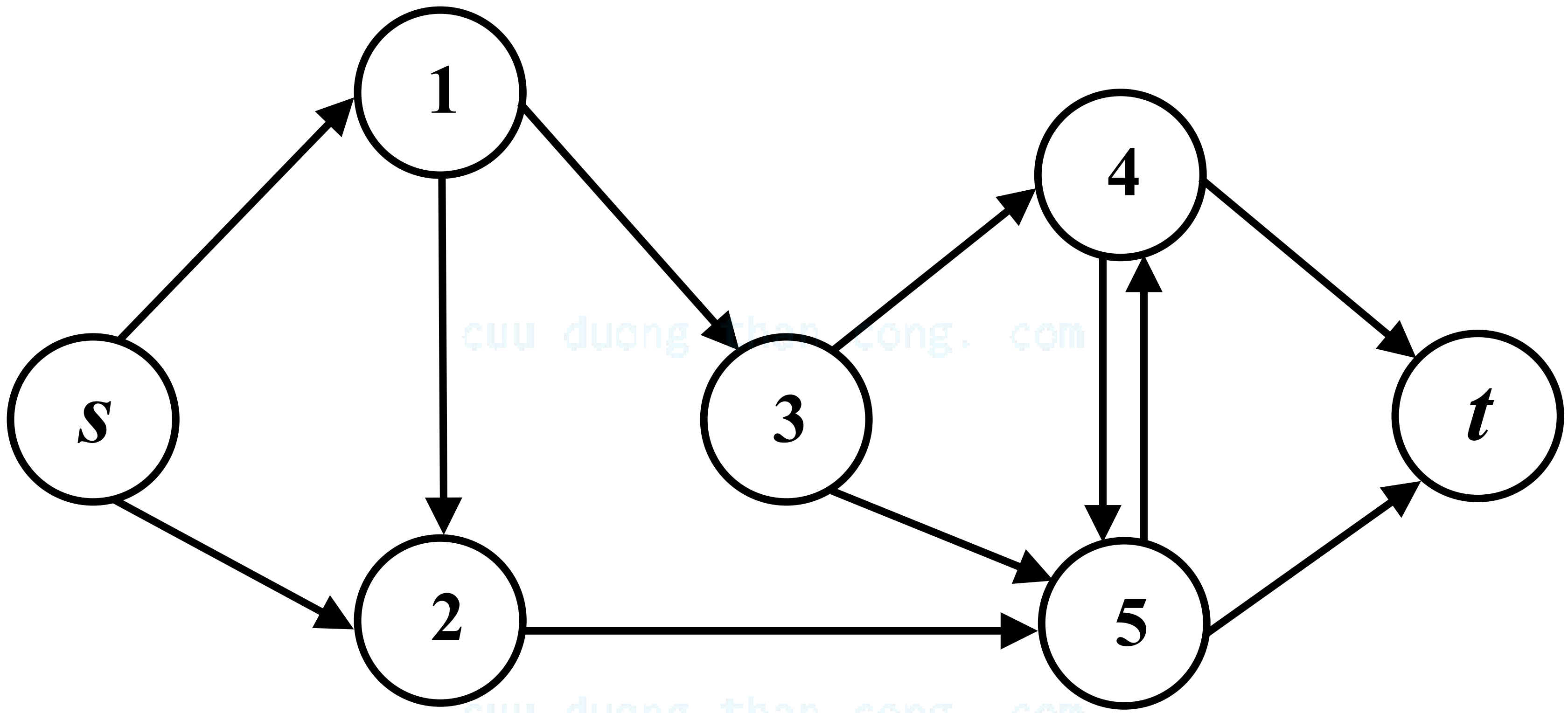
# FLOW NETWORK EXAMPLE

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# TRANSSHIPMENT PROBLEMS

- nodes 1, 2, 3, 4, and 5 are the transshipment points
- arcs of the network are ( $s$, 1 ), ( $s$, 2 ), ( 1, 2 ), ( 1, 3 ), ( 2, 5 ), ( 3, 4 ), ( 3, 5 ), ( 4, 5 ), ( 5, 4 ), ( 4, $t$ ), ( 5, $t$ ) ; the existence of an arc from 4 to 5 and from 5 to 4 allows bidirectional flows between the two nodes
- each arc may be constrained in terms of a limit on the flow through the arc

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# MAX FLOW PROBLEM

- We denote by $f_{ij}$ the flow from $i$ to $j$ and this equals the amount of the commodity shipped from $i$ to $j$ on an arc $(i , j)$ that directly connects the nodes $i$ and $j$
- The problem is to determine the maximal flow $f$ from $s$ to $t$ taking into account the flow limits $k_{ij}$ of each arc $(i , j)$
- The mathematical statement of the problem is

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# MAX FLOW PROBLEM

$$\max \quad Z = f$$

*s.t.*

$$0 \leq f_{ij} \leq k_{ij} \qquad \forall \text{ arc } (i,j) \text{ that connects nodes } i \text{ and } j$$

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$$\left.\begin{array}{lll} f = \sum_i f_{si} & & \text{at source } s \\ \sum_i f_{it} = f & & \text{at sink } t \end{array}\right\} \text{conservation of flow relations}$$

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$$\left.\sum_i f_{ij} = \sum_k f_{jk} \quad \right\} \text{at each transshipment node } j$$



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# MAX FLOW PROBLEM

- While the simplex approach can solve the *max flow* problem, it is possible to construct a highly efficient *network* method to find $f$ directly
- We develop such a scheme by making use of network or graph theoretic notions
- We start by introducing some definitions

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# DEFINITIONS OF NETWORK TERMS

- Each *arc* is directed and so for an arc $(i\ ,j)$, $f_{ij} \geq 0$

- A *forward* arc at a node $i$ is one that leaves the node $i$ to some node $j$ and is denoted by $(i\ ,j)$

- A *backward* arc at node $i$ is one that enters node $i$ from some node $j$ and is denoted by $(j\ ,i)$

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# DEFINITIONS OF NETWORK TERMS

- A *path* connecting node $i$ to node $j$ is a sequence of arcs that starts at node $i$ and terminates at node $j$
  - we denote a path by

$$\mathscr{P} = \{ (i, k), (k, l), \ldots, (m, j) \}$$

  - in the example network
    - ( 1, 2 ), ( 2, 5 ), ( 5, 4 ) is a path from 1 to 4
    - ( 1, 3 ), ( 3, 4 ) is another path from 1 to 4

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# DEFINITIONS OF NETWORK TERMS

- A *cycle* is a path with $i = j$, i.e.,

$$\mathscr{P} = \{ (i, k), (k, l), \ldots, (m, i) \}$$

- We denote the set of nodes of the network by $\mathscr{N}$

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  - the definition is

$$\mathscr{N} = \{ i : i \text{ is a node of the network} \}$$

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  - In the example network

$$\mathscr{N} = \{ s, 1, 2, 3, 4, 5, t \}$$

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# NETWORK CUT

- A *cut* is a partitioning of nodes into two distinct subsets $\mathscr{S}$ and $\mathscr{T}$ with

$$\mathscr{N} = \mathscr{S} \cup \mathscr{T} \text{ and } \mathscr{S} \cap \mathscr{T} = \varnothing$$

- We are interested in cuts with the property that

$$s \in \mathscr{S} \text{ and } t \in \mathscr{T}$$

  - the sets $\mathscr{S}$ and $\mathscr{T}$ provide an $s - t$ cut
  - in the example network,

$$\mathscr{S} = \{ s, 1, 2 \} \text{ and } \mathscr{T} = \{ 3, 4, 5, t \}$$

  provide an $s - t$ cut

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# NETWORK CUT

- The capacity of a cut is

$$K(\mathscr{S},\mathscr{T}) = \sum_{\substack{i \in \mathscr{S} \\ j \in \mathscr{T}}} k_{ij}$$

- In the example network with

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$$\mathscr{S} = \{ s, 1, 2 \} \quad \text{and} \quad \mathscr{T} = \{ 3, 4, 5, t \}$$

we have

$$K(\mathscr{S},\mathscr{T}) = k_{13} + k_{25}$$

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but for the cut with

$$\mathscr{S} = \{s, 1, 2, 3, 4 \} \quad \text{and} \quad \mathscr{T} = \{ 5, t \}$$

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# NETWORK CUT

$$K(\mathscr{S},\mathscr{T}) = k_{4,t} + k_{4,5} + k_{3,5} + k_{2,5}$$

- Now, arc (5, 4) is directed from a node in $\mathscr{T}$ to a node in $\mathscr{S}$ and is not included in the summation
- An important characteristic of the $s - t$ cuts of interest is that if all the arcs in the cut are removed, then *no* path exists from $s$ to $t$ ; consequently, no flow is possible since any flow from $s$ to $t$ must go through the arcs in a cut
- The flow is *limited* by the capacity of the cut

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# NETWORK CUT LEMMA

- For any directed network, the flow $f$ from $s$ to $t$ is constrained by an $s-t$ cut

$$f \leq K\ (\mathscr{S},\mathscr{T})\ \textit{for any cut}\ \mathscr{S},\mathscr{T}$$

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- Corollaries of this lemma are

$$\textit{(i)}\ \ max\ flow \leq K\ (\mathscr{S},\mathscr{T})\ \ \forall\ \mathscr{S},\mathscr{T}$$

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and

$$\textit{(ii)}\ \ max\ flow \leq \min_{\mathscr{S},\mathscr{T}}\ K\ (\mathscr{S},\mathscr{T})$$

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# MAX – FLOW – MIN – CUT THEOREM

- For any network, the value of the maximal flow from $s$ to $t$ is equal to the minimal cut, i.e., the cut $\mathscr{S}$, $\mathscr{T}$ with the smallest capacity

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- The *max-flow min-cut* theorem allows us, in principle, to find the maximal flow in a network by finding the capacities of all the cuts and determining the cut with the smallest capacity

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# MAX FLOW

- The *maximal flow* algorithm is based on finding a *path* through which a positive flow from $s$ to $t$ can be sent, the so – called *flow augmenting path*; the procedure is continued until no such flow augmenting path can be found and therefore we have the *maximal flow*
- The maximal flow algorithm is based on the repeated application of the *labeling procedure*

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# LABELING PROCEDURE

- The *labeling procedure* is used to find a *flow augmenting path* from $s - t$

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- We say that a node $j$ can be labeled if and only if flow can be sent from $s$ to $t$

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# LABELING PROCEDURE

**Step** $0$ **: start with node** $s$

**Step** $1$ **: label node** $j$ **given that node** $i$ **is labeled only if**

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($i$) **either there exists an arc** $( i , j )$ **and**

$$f_{ij} < k_{ij}$$

($ii$) **or there exists an arc** $( j , i )$ **and**

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$$f_{ji} > 0$$

**Step** $2$ **: if** $j = t$ **, stop; else, go to Step 1**

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# THE MAX FLOW ALGORITHM

**Step *0* : start with a feasible flow**

**Step 1 : use the *labeling procedure* to find a flow augmenting path**

**Step 2 : determine the maximum value $\delta$ for the max increase (decrease) of flow on all forward (backward) arcs**

**Step 3 : use the *labeling procedure* to find a flow augmenting path; if no such path exists, stop and go to Step 2**

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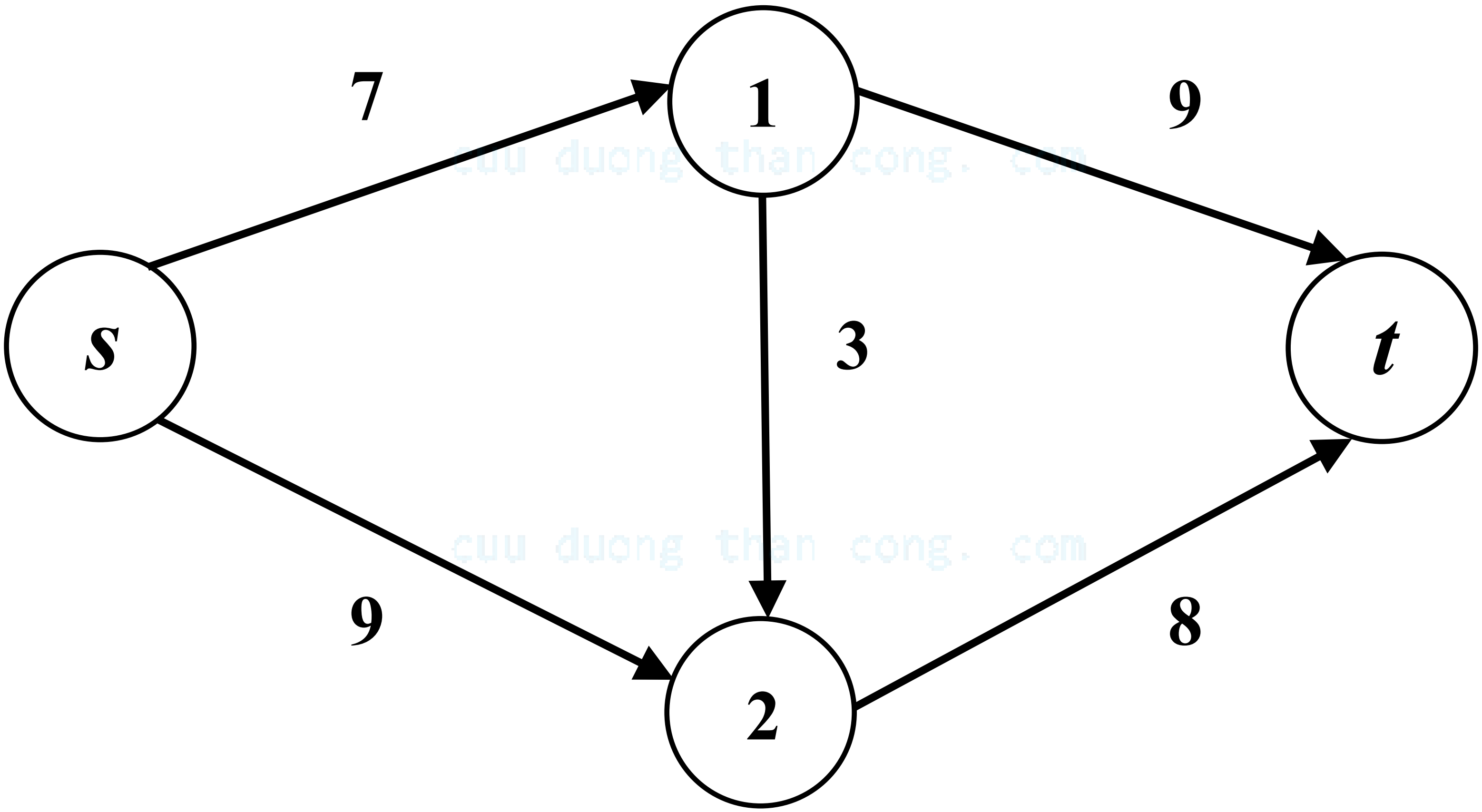
# EXAMPLE

- Consider the simple network with the flow capacities on each arc indicated

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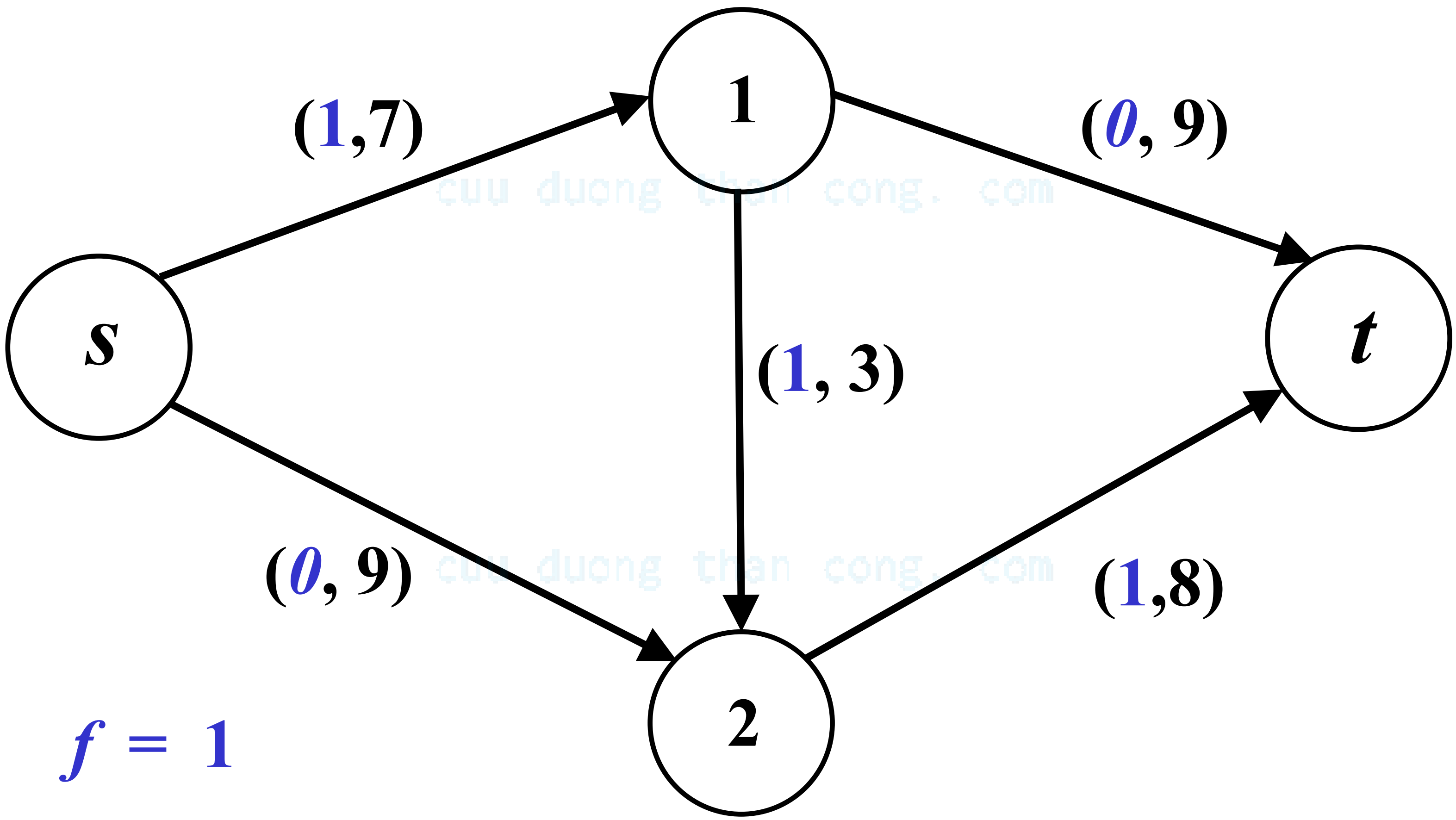
# EXAMPLE

- We initialize the network with a flow 1




$f = 1$

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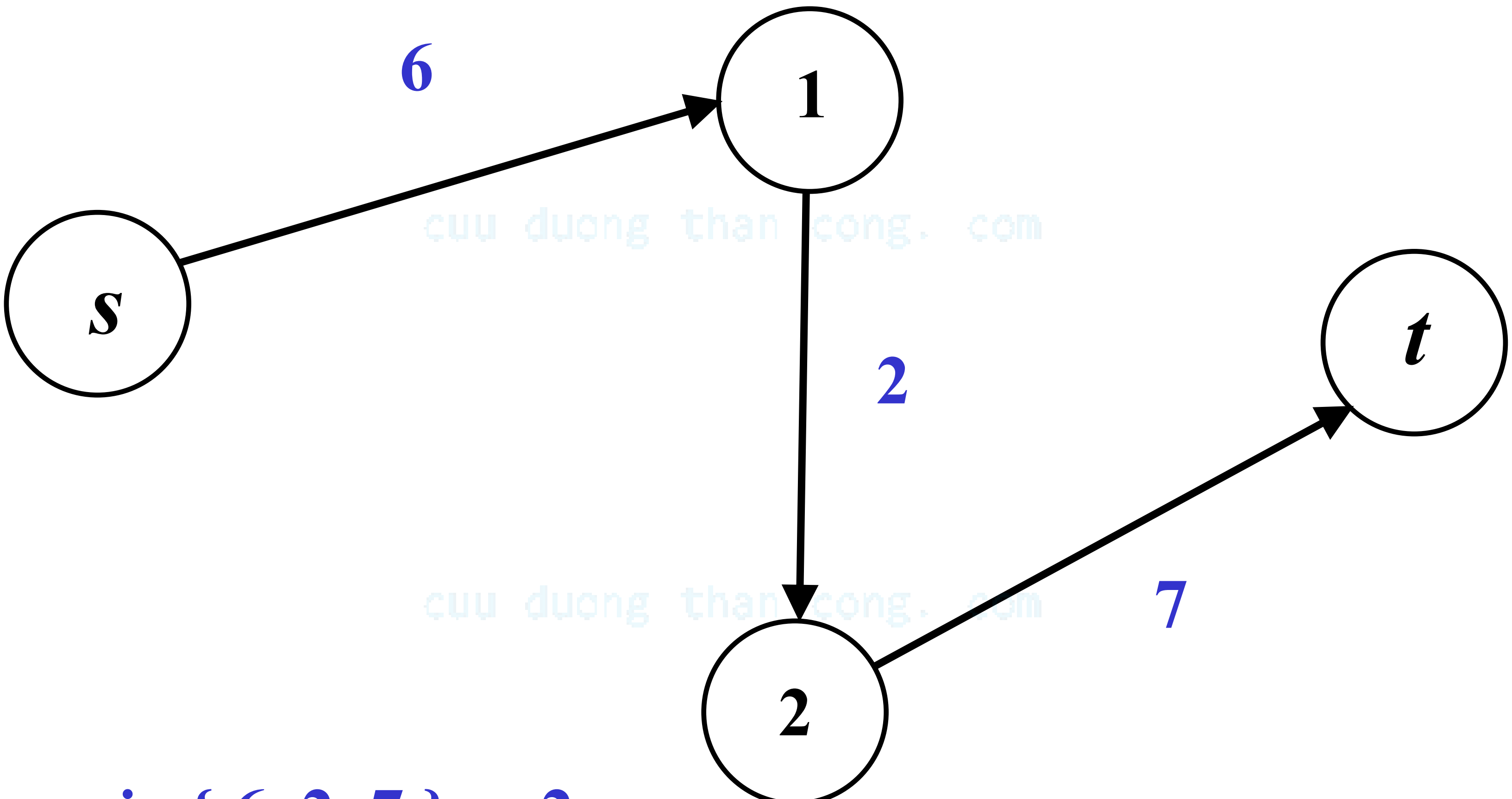
# EXAMPLE

- We apply the labeling procedure




$f = \min \{ 6, 2, 7 \} = 2$

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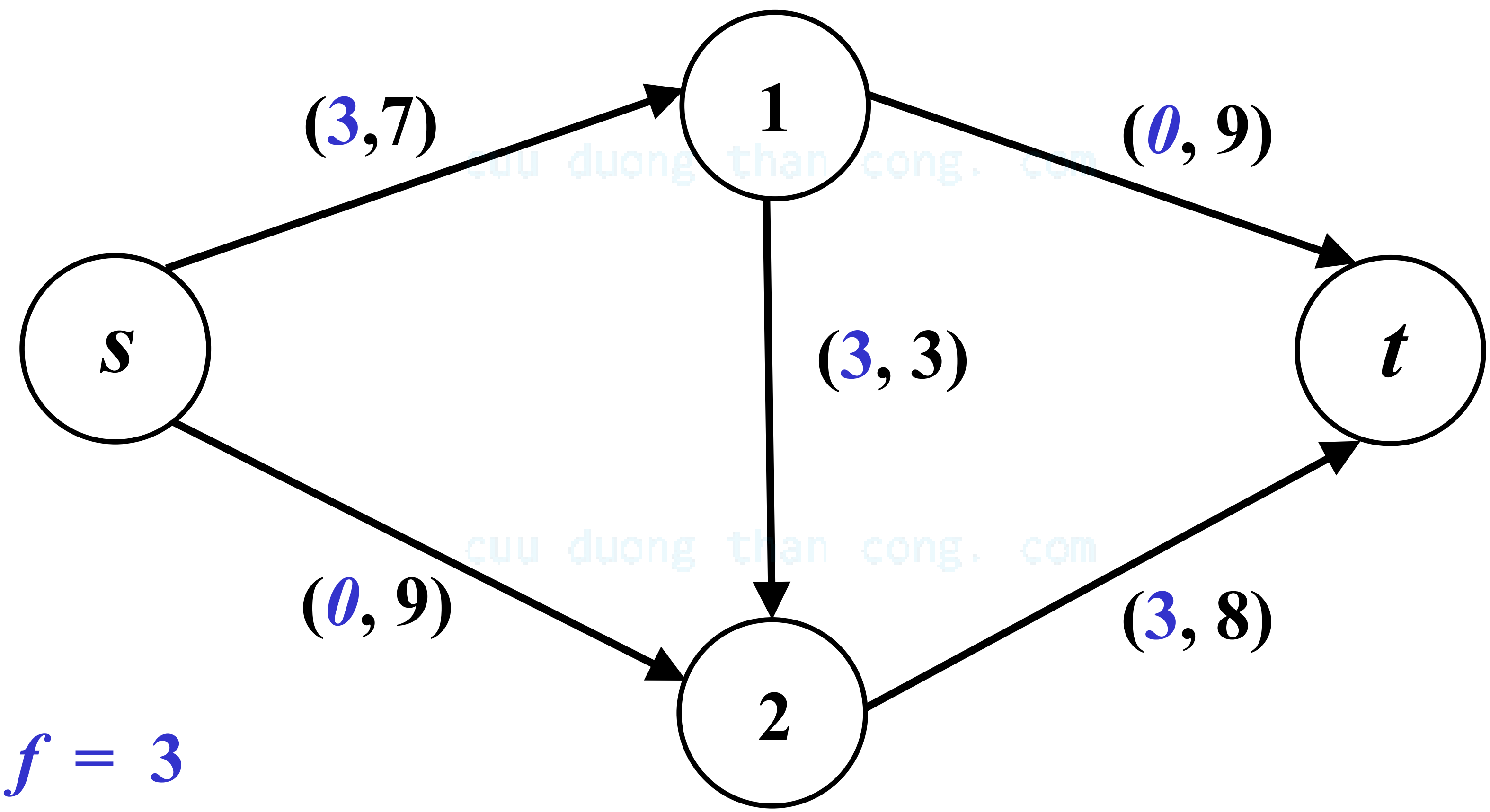
# EXAMPLE

- Consider the simple network with the flow and the capacity on each arc $(i, j)$ indicated by $(f_{ij}, k_{ij})$




$f = 3$



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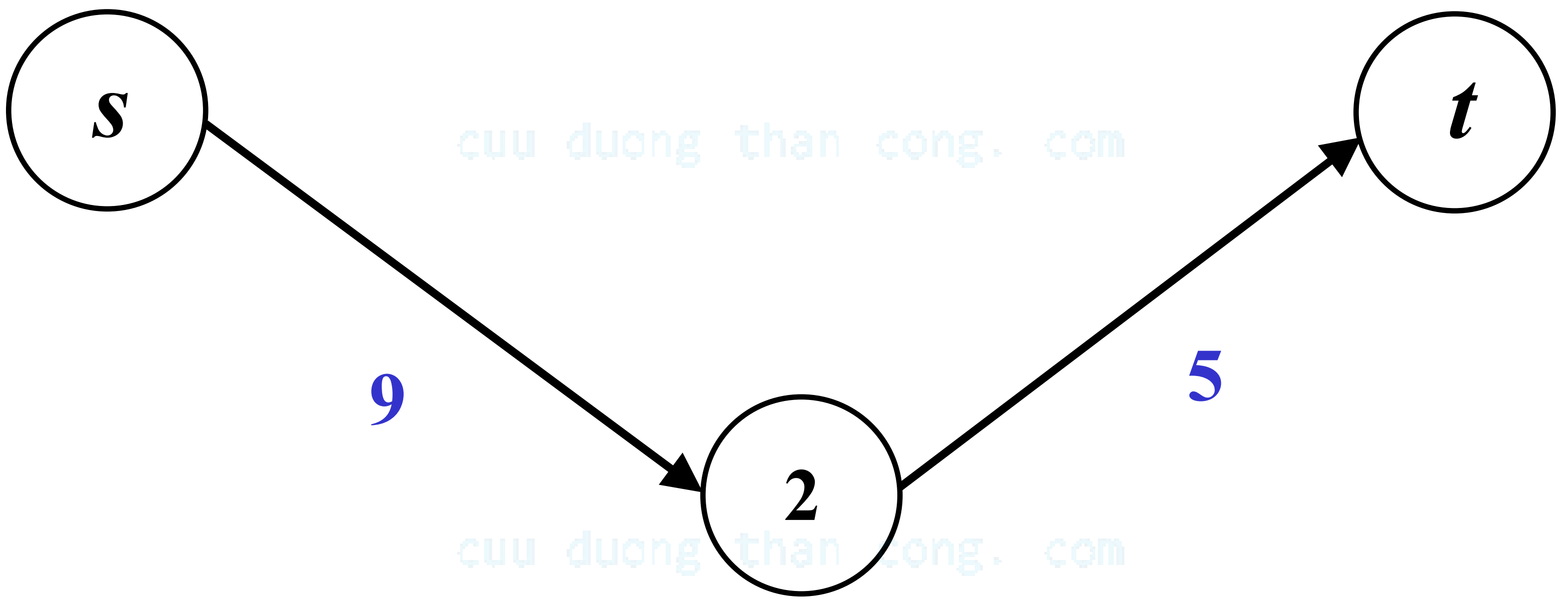
# EXAMPLE

- We repeat application of the labeling procedure




$$f = min\{5, 9\} = 5$$

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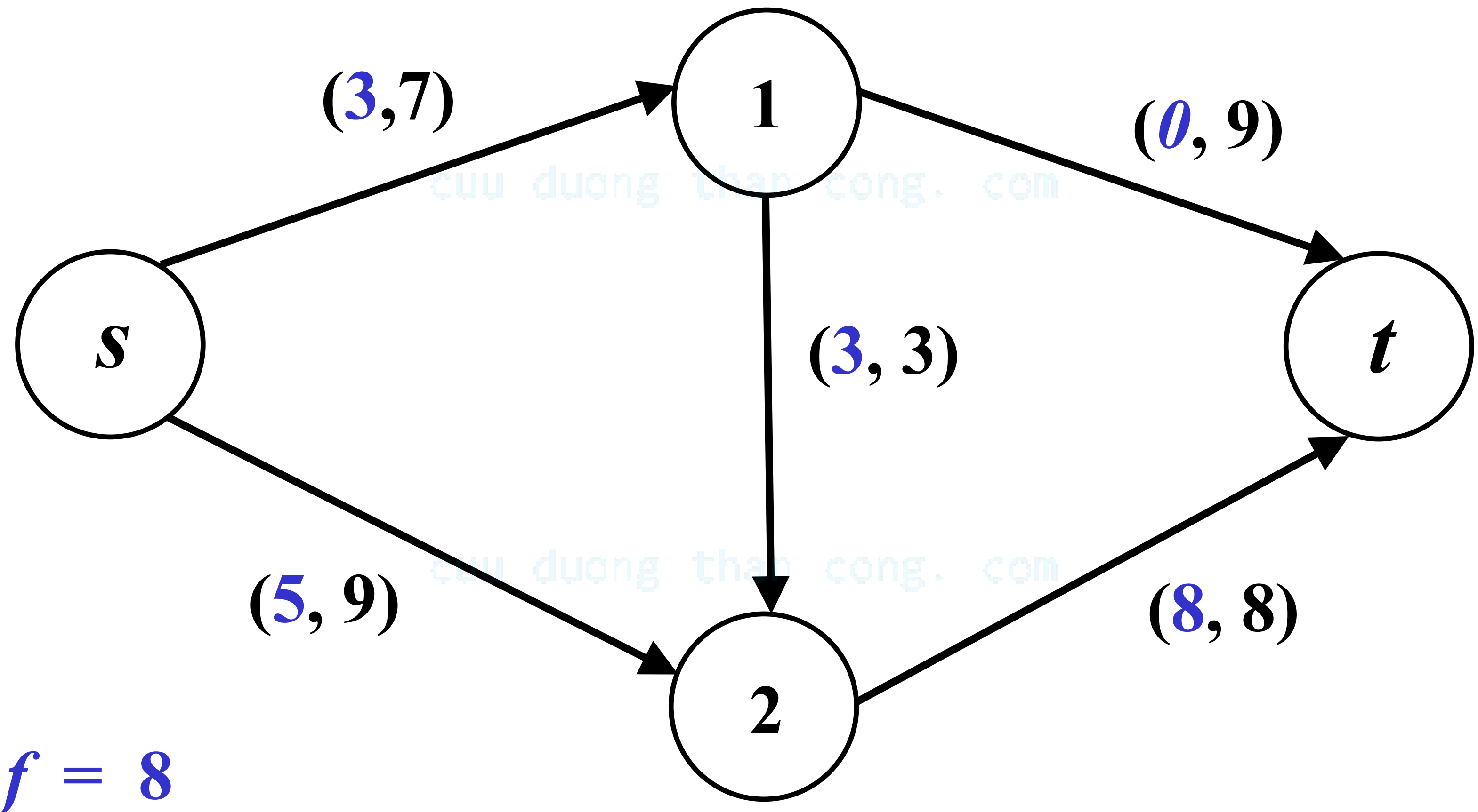
# EXAMPLE

- We increase the flow by 5




$f = 8$

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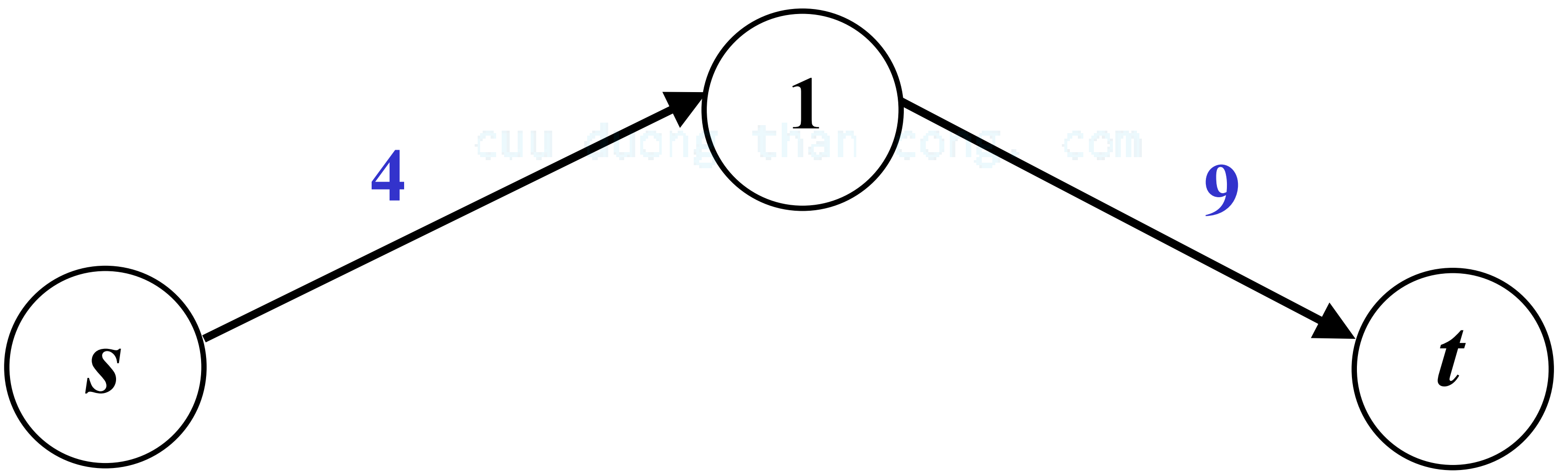
# EXAMPLE

- We repeat application of the labeling procedure




$f = min \{ 4, 9 \} = 4$

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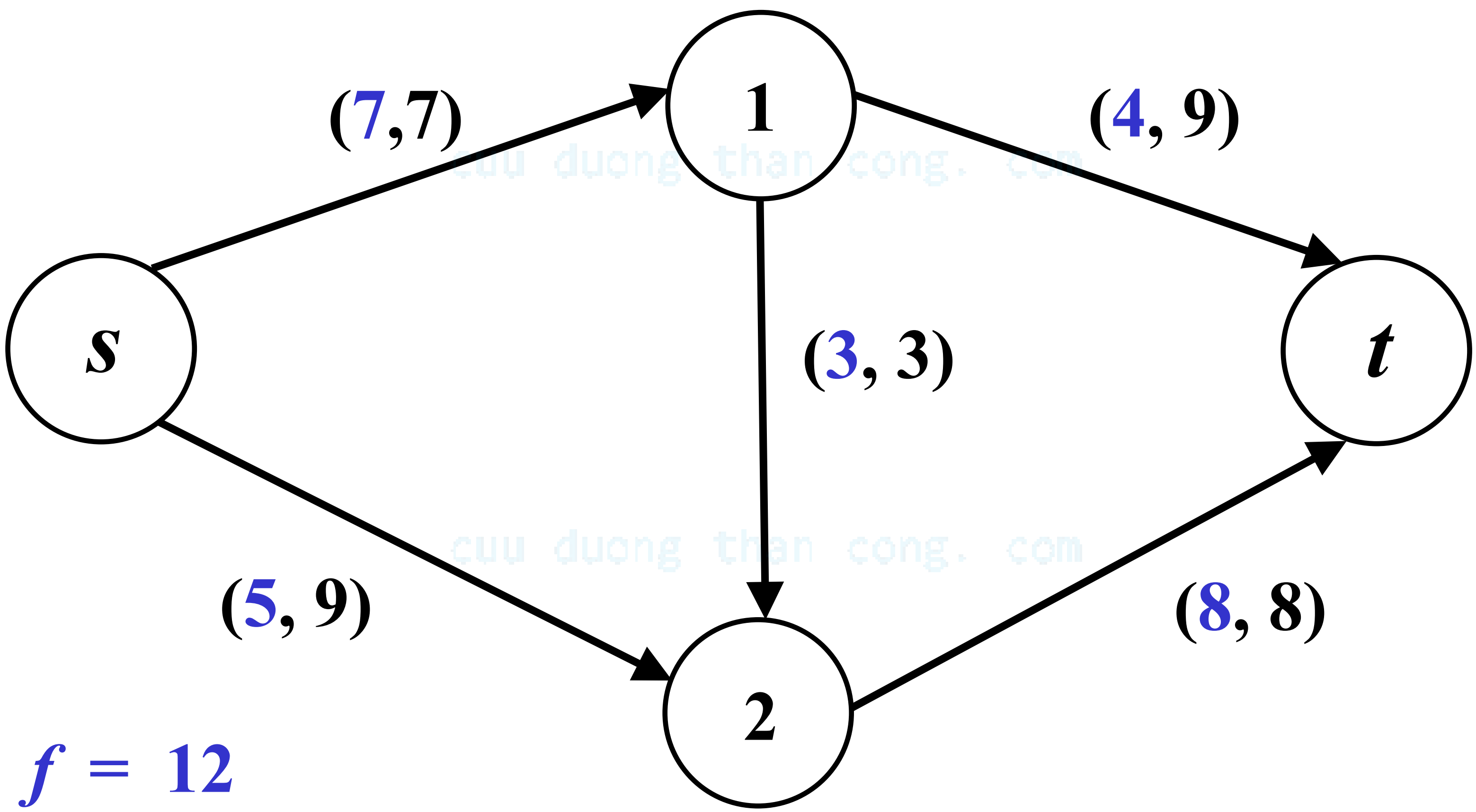
# EXAMPLE

- We increase the flow by 4 to obtain




$f = 12$

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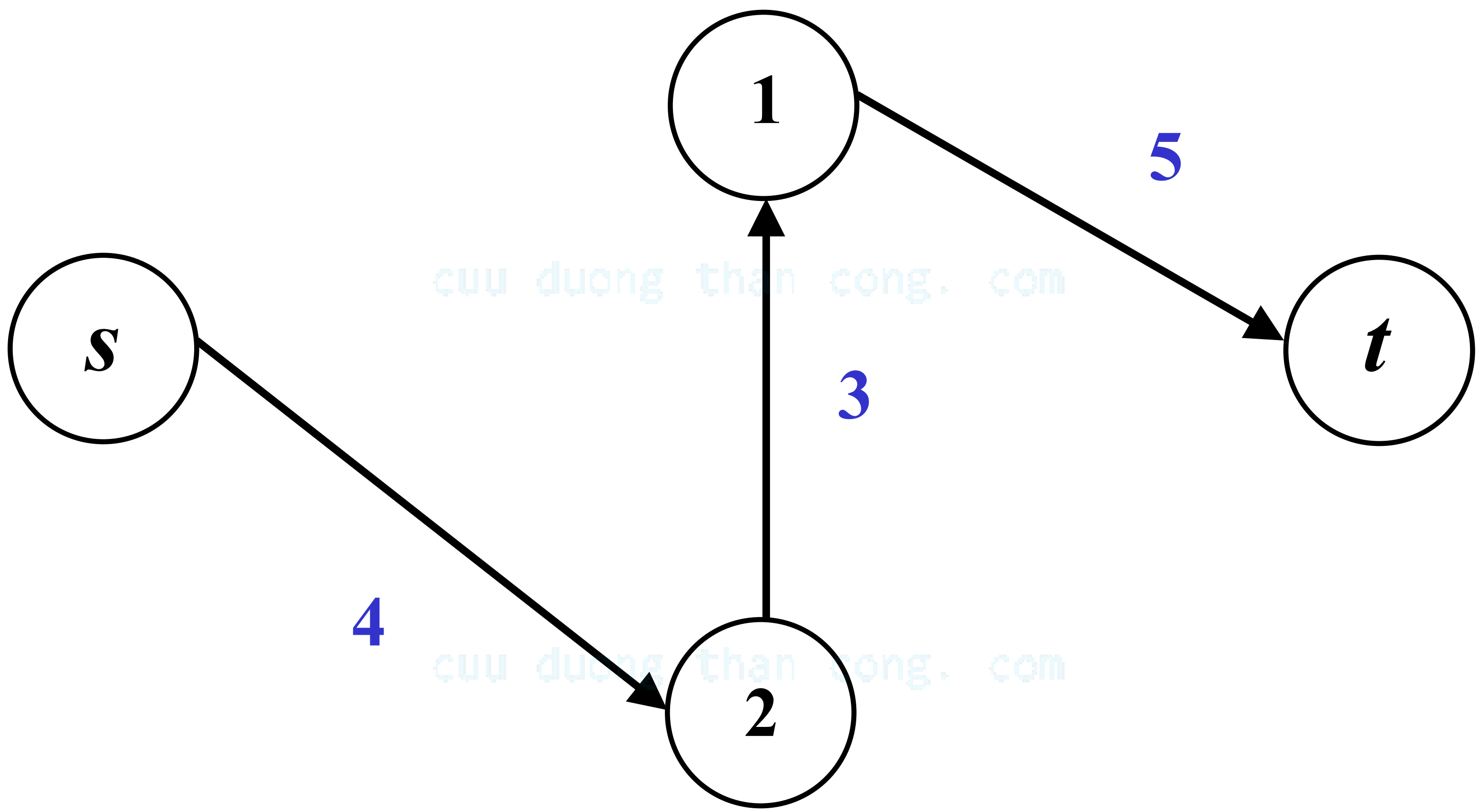
# EXAMPLE

- We repeat application of the *labeling procedure*




$$f = min \{ 4, 3, 5 \} = 3$$

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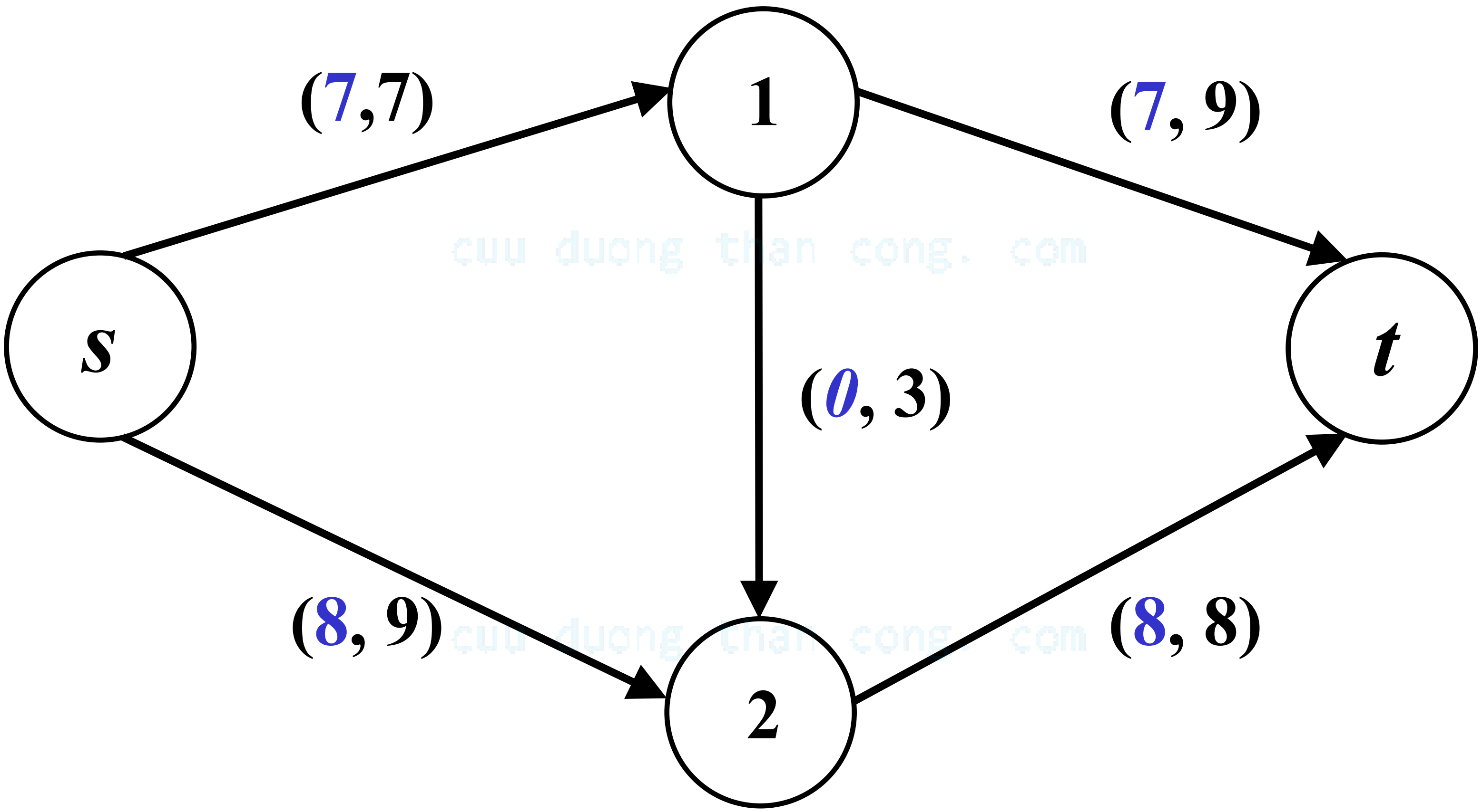
# EXAMPLE

- We increase the flow by 3




$f$ = 15 with no possibility of a *flow augmenting path*

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# UNDIRECTED NETWORKS

- A network with undirected arcs is called an undirected network: the flows on each arc $( i, j )$ with the limit $k_{ij}$ cannot violate the capacity constraints in either direction
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- Mathematically, we require

$$\left.\begin{aligned} f_{ij} &\leq k_{ij} \\ f_{ji} &\leq k_{ji} \\ f_{ij} f_{ji} &= 0 \end{aligned}\right\}$$

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interpretation of unidirectional flow below capacity limit

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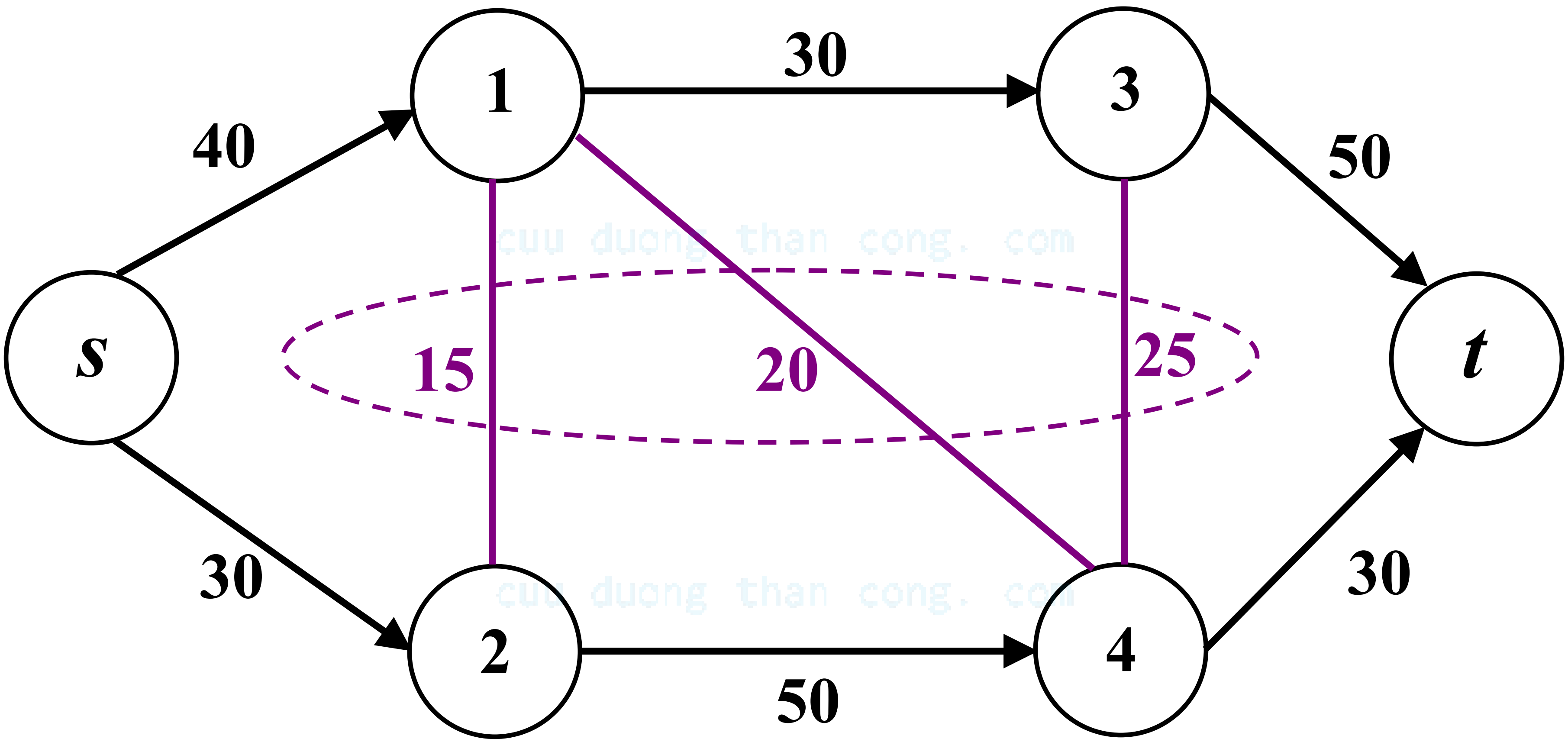
# EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS

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# EXAMPLE OF A NETWORK WITH UNDIRECTED ARCS

- To make the problem realistic, we may view the capacities as representing traffic flow limits: the directed arcs correspond to *unidirectional* streets and the problem is to place *one-way signs* on each street $(i, j)$ not already directed, so as to *maximize* traffic flow from $s$ to $t$
- The procedure is to replace each *undirected arc* by two *directed* arcs $(i, j)$ and $(j, i)$ to deter-mine the maximal $s - t$ flow

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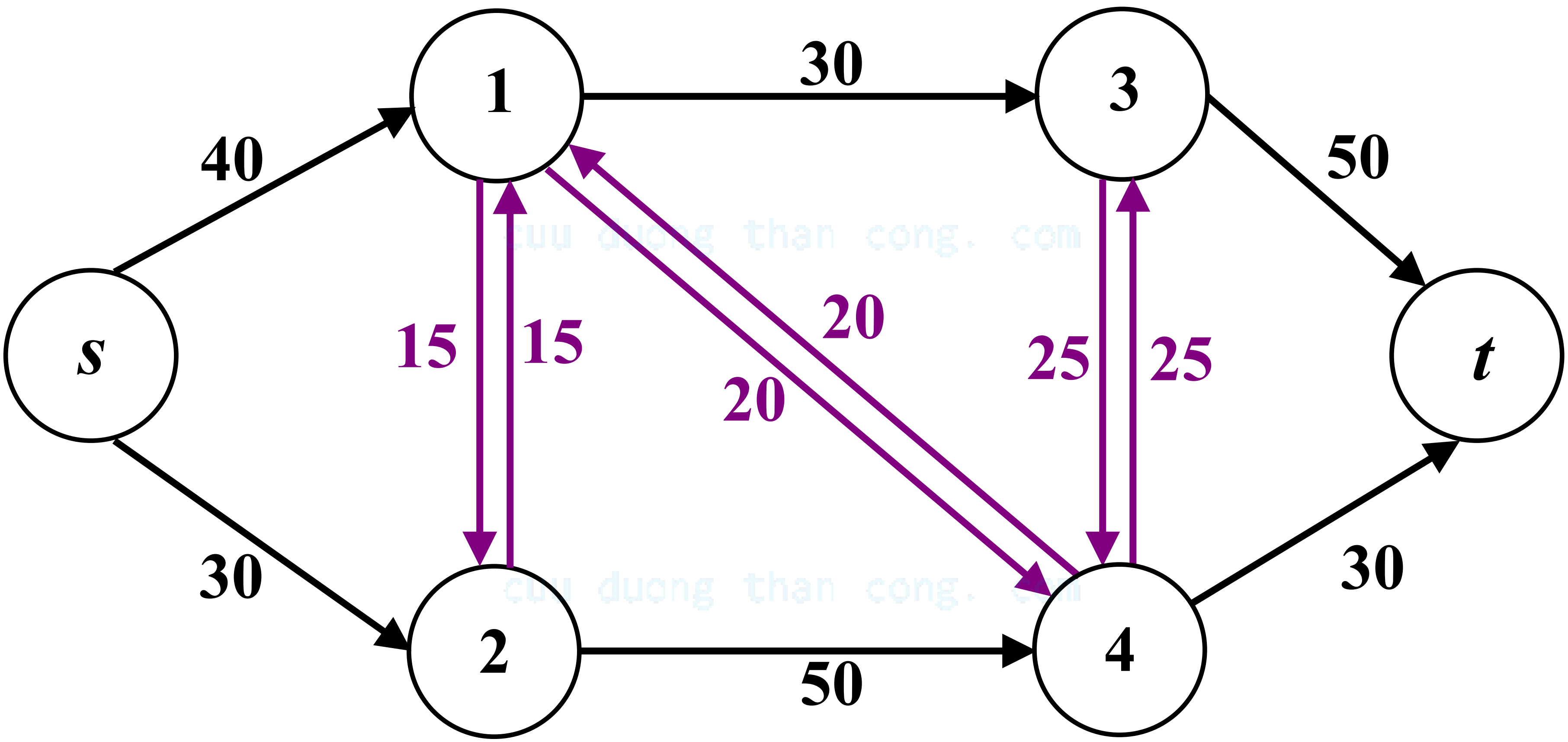
# EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS

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# EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS

- We apply the *max flow* scheme to the directed network and give the following interpretations to the flows on the *max flow* bidirectional arcs that are the initially undirected arcs $(i, j)$ : if

$$f_{ij} > 0 \ , \ f_{ji} > 0 \ \text{and} \ f_{ij} > f_{ji}$$

  set up the flow from $i$ to $j$ with value $f_{ij} - f_{ji}$ and remove the arc $(j, i)$

- The computation of the max flow $f$ for this example is left as a homework exercise

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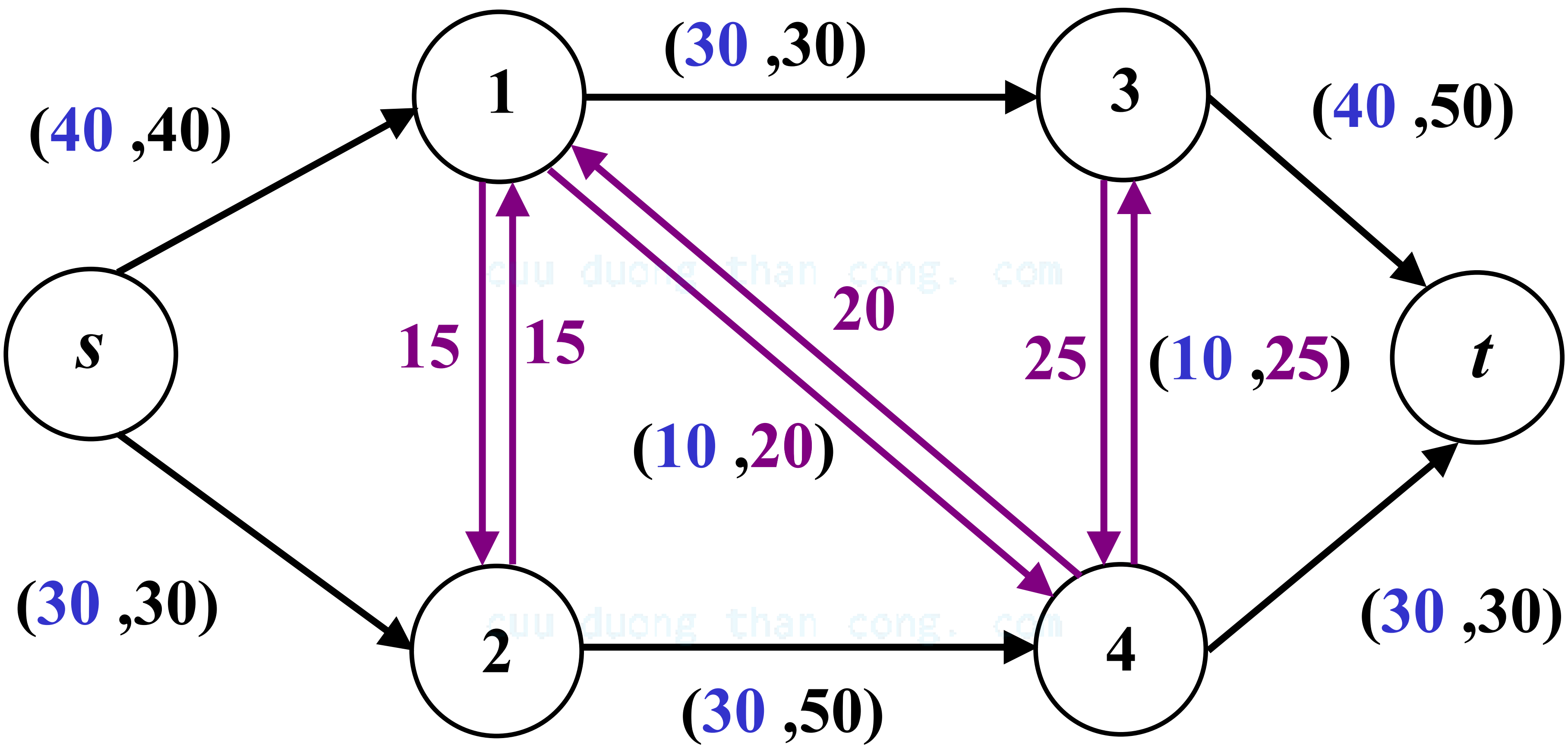
# EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS

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# EXAMPLE OF A NETWORK WITH 3 UNDIRECTED ARCS : RESULT

flow: $s \rightarrow 1 \rightarrow 3 \rightarrow t \quad = \quad 30$

flow: $s \rightarrow 2 \rightarrow 4 \rightarrow t \quad = \quad 30$

flow: $s \rightarrow 1 \rightarrow 4 \rightarrow 3 \rightarrow t \quad = \quad 10$

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and so the maximum flow is $30 + 30 + 10 = 70$

one way signs should be put from $1 \rightarrow 4$ and $4 \rightarrow 3$

an alternative routing of a flow of 10 is the path

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$s \rightarrow 1 \rightarrow 2 \rightarrow 4 \rightarrow 3 \rightarrow t$ which would require one way signs from $1 \rightarrow 2$ and $4 \rightarrow 3$

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# NETWORKS WITH MULTIPLE SOURCES AND MULTIPLE SINKS

- **We next consider a network with several supply and several demand points**

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- **We introduce a super source $\hat{s}$ linking to all the sources and a super sink $\hat{t}$ linking all the sinks**

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- **We can consequently apply the *max flow* algorithm to the modified network**

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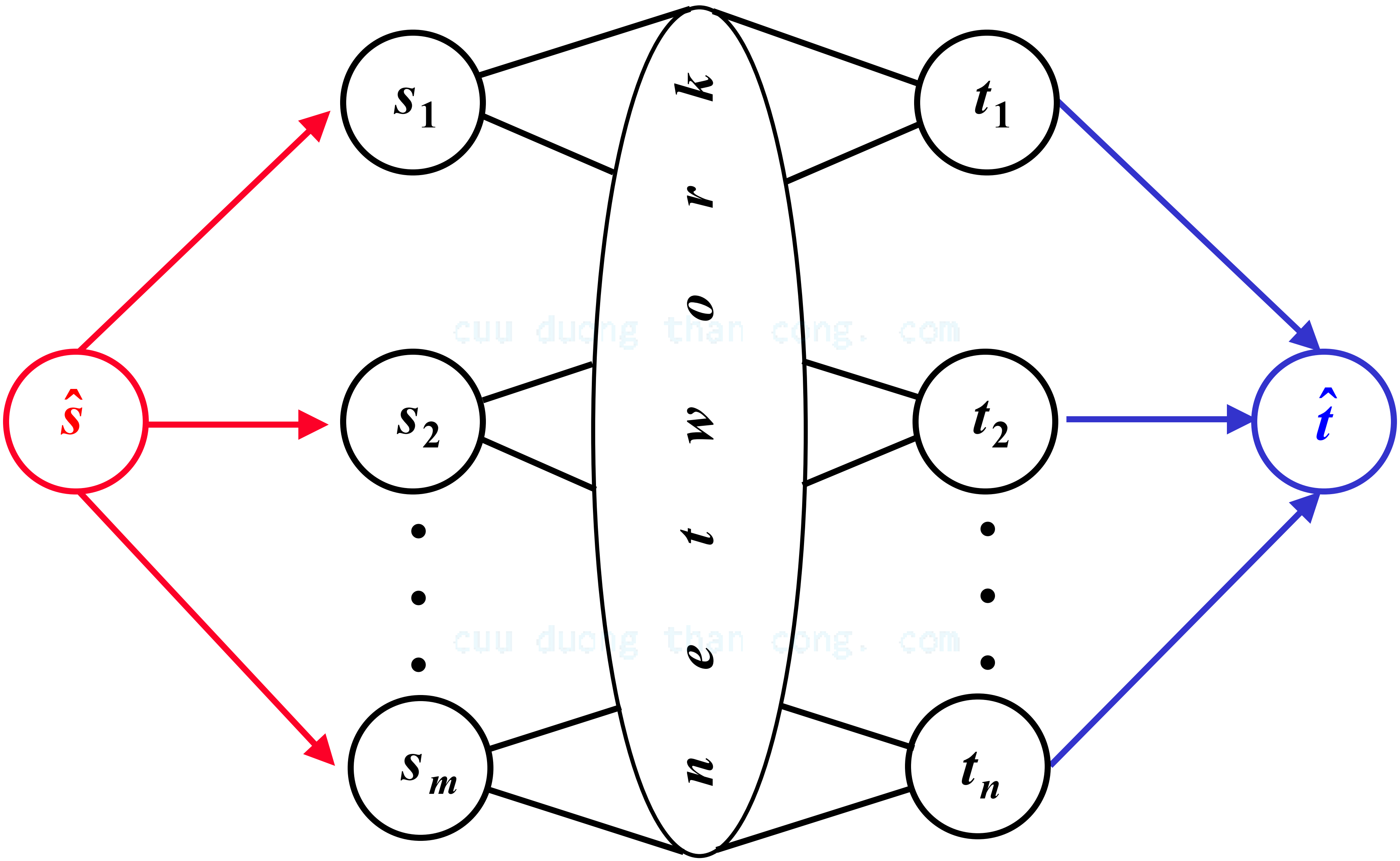
# NETWORKS WITH MULTIPLE SOURCES AND MULTIPLE SINKS

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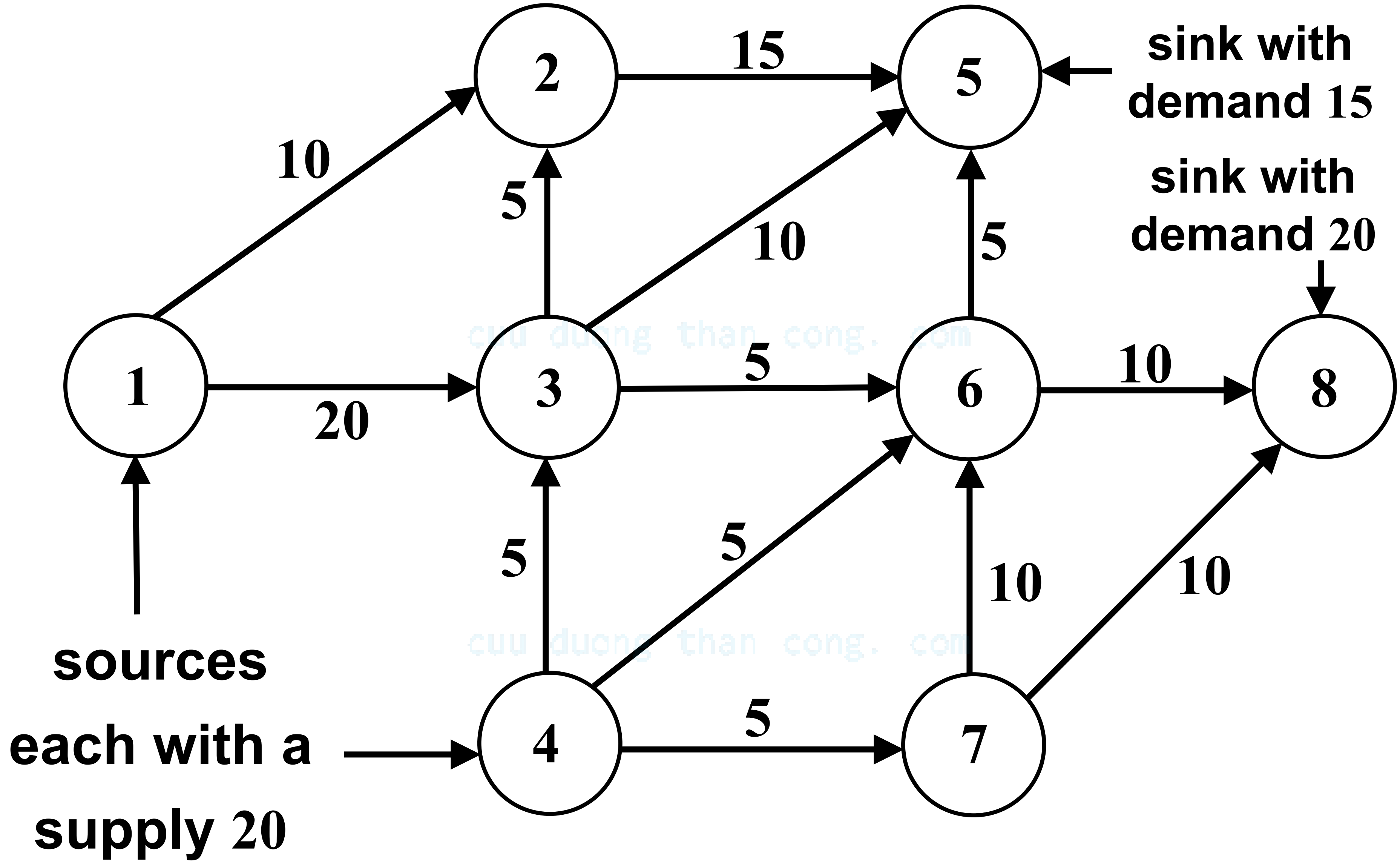
# MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

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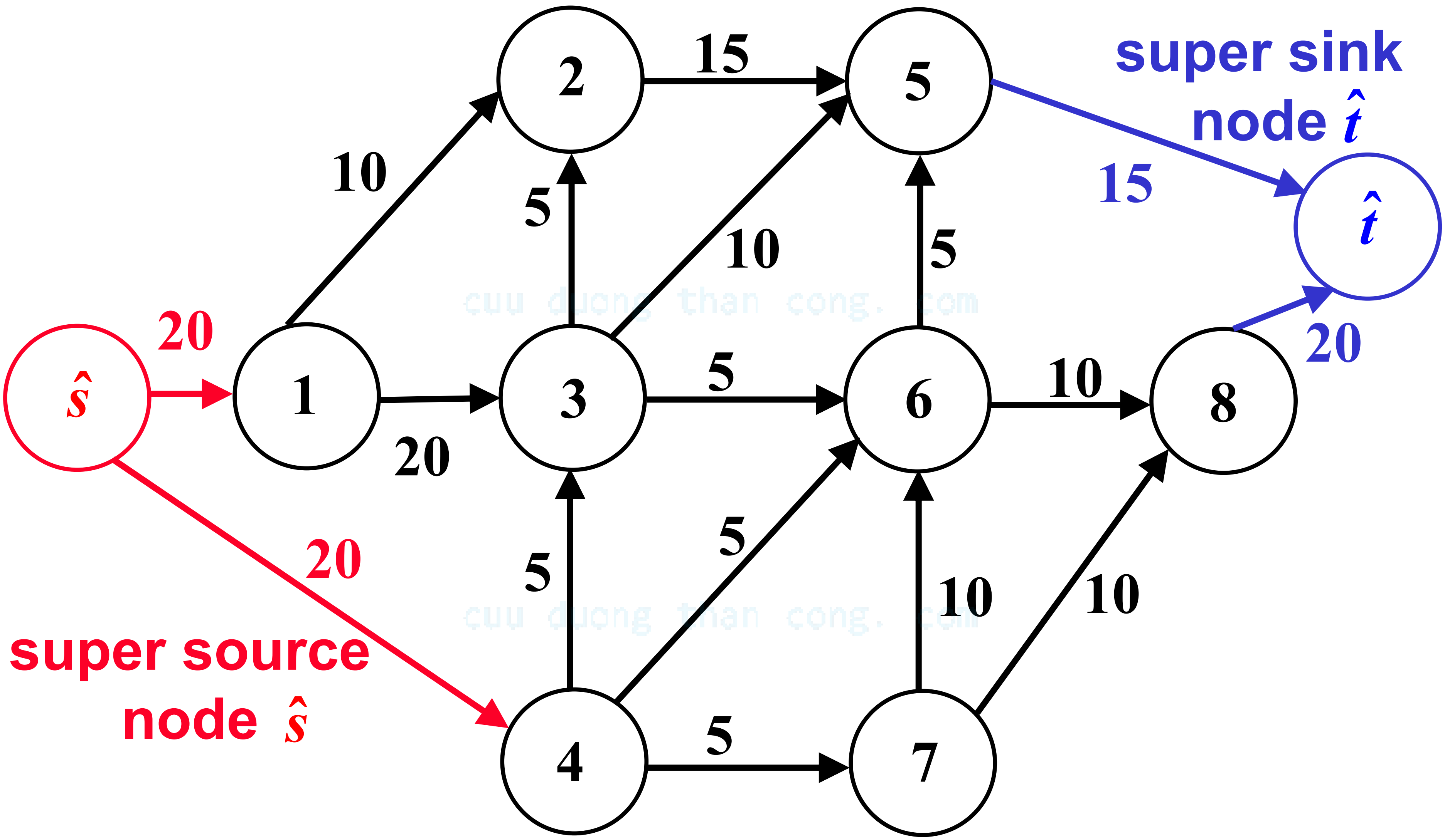
# MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

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# MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

- The transshipment problem is feasible if and only if the maximal $\hat{s}-\hat{t}$ flow $f$ satisfies

$$f = \sum_{sinks} demands$$

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- We need to show that
  - the transshipment problem is infeasible since the network cannot accommodate the total demand of $35$ ;
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  - the smallest shortage for this problem is $5$

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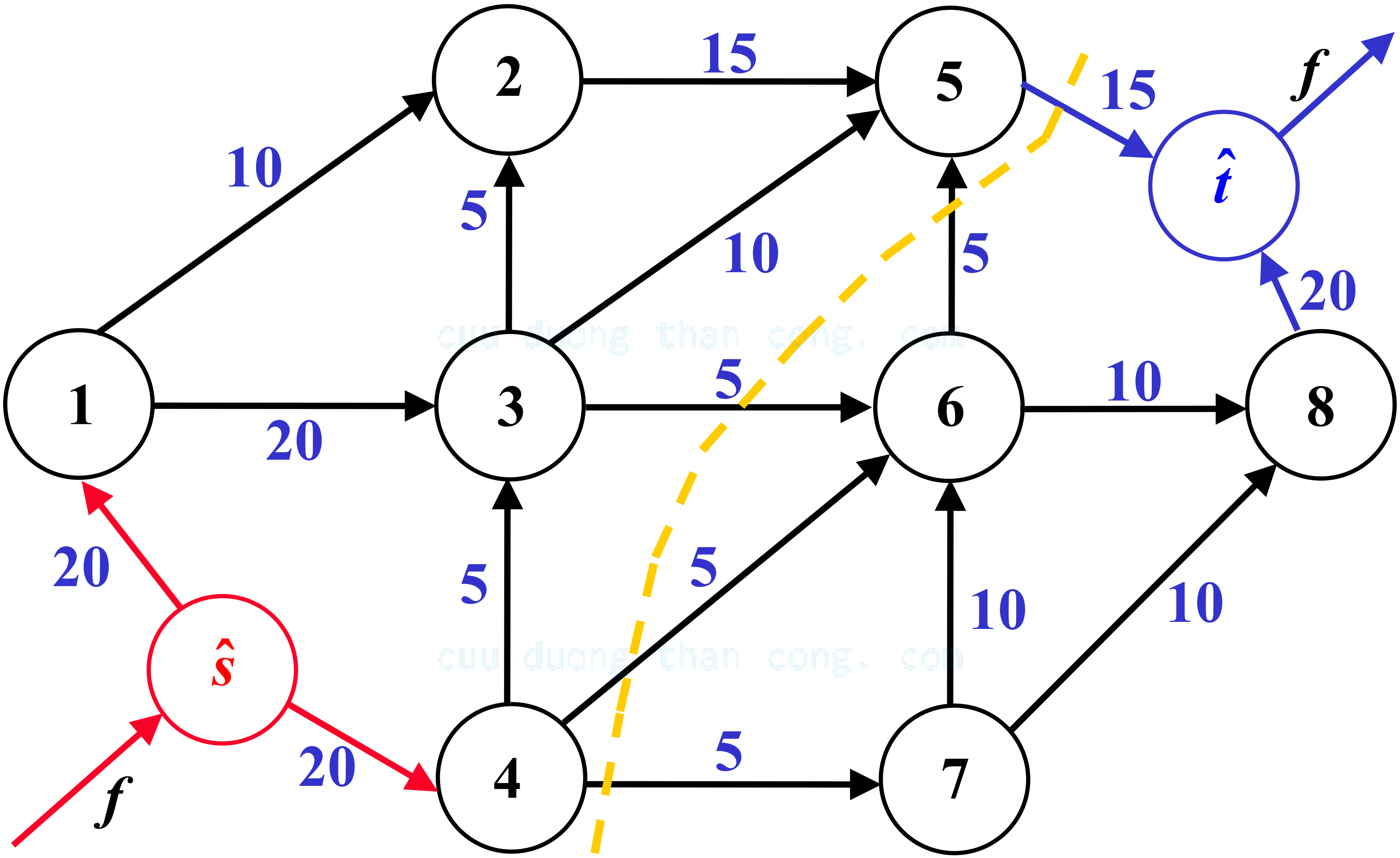
# MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

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# MULTIPLE – SOURCE / MULTIPLE – SINK NETWORK EXAMPLE

- The minimum cut is shown and has capacity

$$15 + 5 + 5 + 5 = 30$$

  and the maximum flow is, therefore, 30

- Since the maximum flow fails to meet the total demand of 35 units, the problem is infeasible; the minimum shortage is 5

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# APPLICATION TO MANPOWER SCHEDULING

- Consider the case of a company that must complete its four projects in 6 months

| *project* | *earliest start month* | *latest finish month* | *manpower requirements (person/month)* |
|---|---|---|---|
| *A* | 1 | 4 | 6 |
| *B* | 1 | 6 | 8 |
| *C* | 2 | 5 | 3 |
| *D* | 1 | 6 | 4 |

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# APPLICATION TO MANPOWER SCHEDULING

- There are the following additional constraints:
  - the company has only 4 engineers
  - at most 2 engineers may be assigned to any one project in a given month
  - no engineer may be assigned to more than one project at any time
- The question is whether there is a *feasible assignment* and if so determine the *optimal assignment*

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# APPLICATION TO MANPOWER SCHEDULING

- The solution approach is to set up the problem as a transshipment network
  - the sources are the $6$ months of duration
  - the sinks are the $4$ projects
  - an arc $(i, j)$ is introduced whenever a feasible assignment of the engineers working in month $i$ can be made to project $j$ with

$$k_{ij} = 2 \qquad i = 1, 2, \ldots, 6 , \quad j = A , B , C , D$$

  - there is no arc $(1, C)$ since project $C$ cannot start before month $2$

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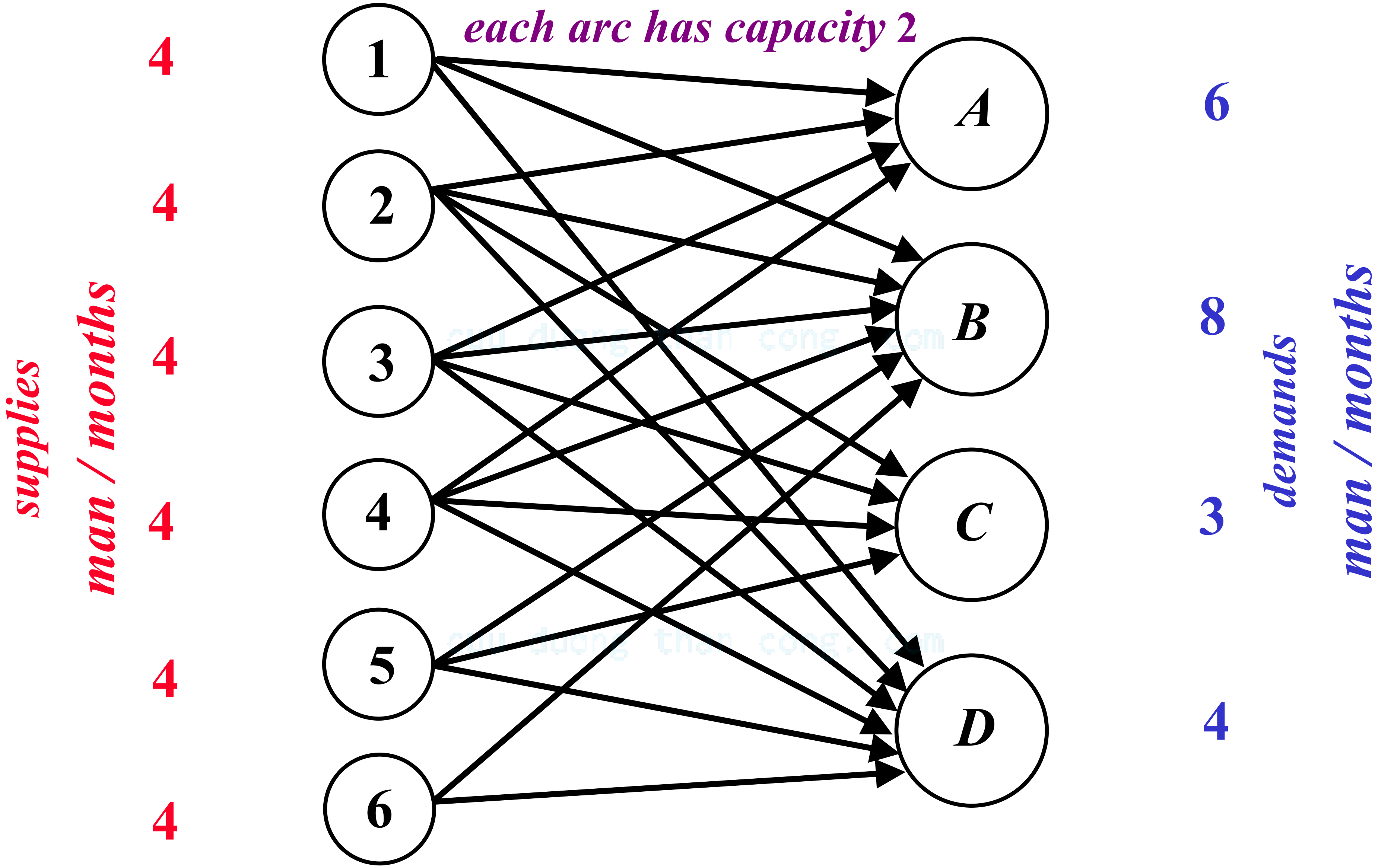
# APPLICATION TO MANPOWER SCHEDULING

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# APPLICATION TO MANPOWER SCHEDULING

- **The transshipment problem is feasible if the total demand**

$$6 + 8 + 3 + 4 = 21$$

  **can be met**

- **As a homework problem, determine whether a feasible schedule exists and if so, find it**

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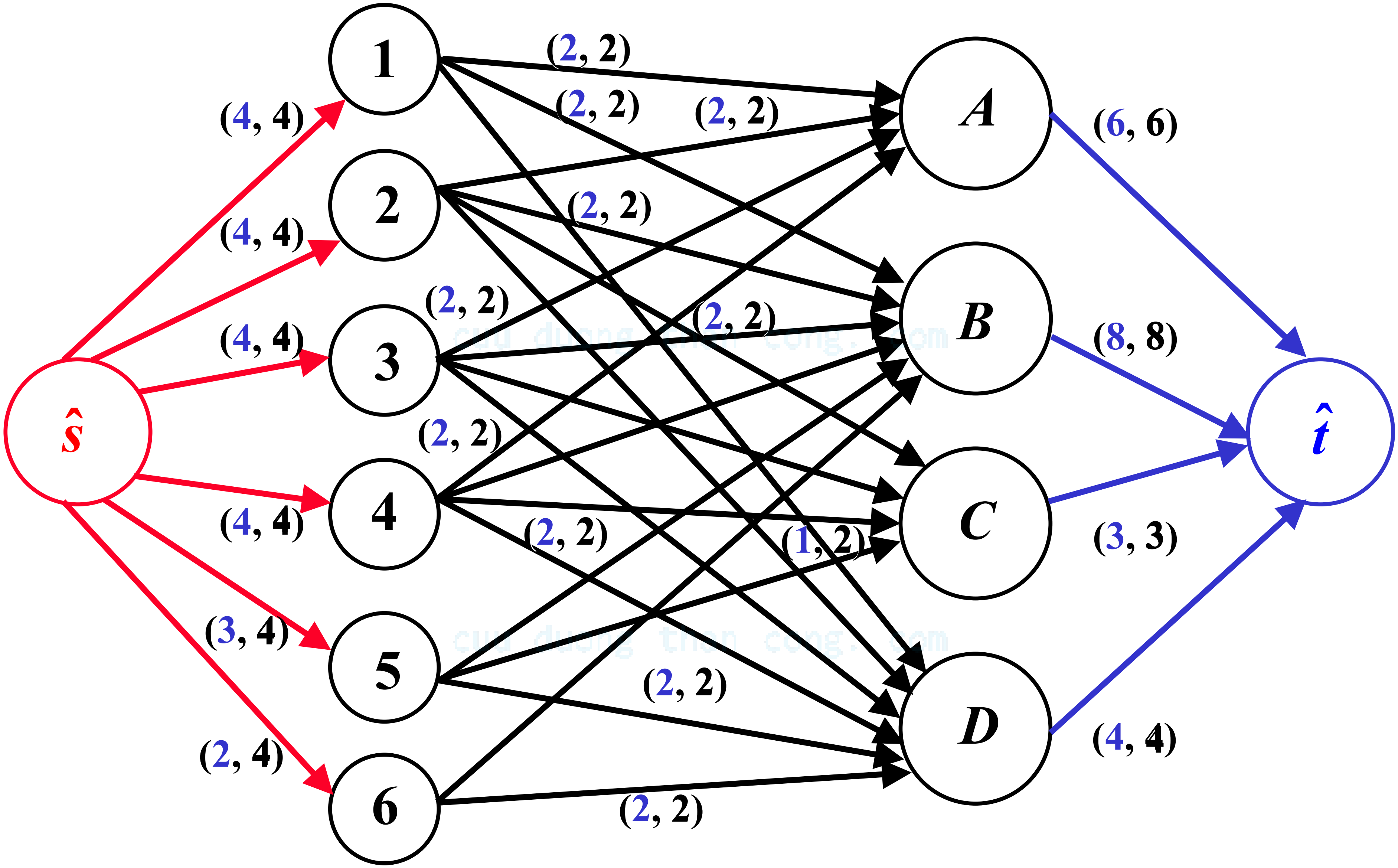
# APPLICATION TO MANPOWER SCHEDULING

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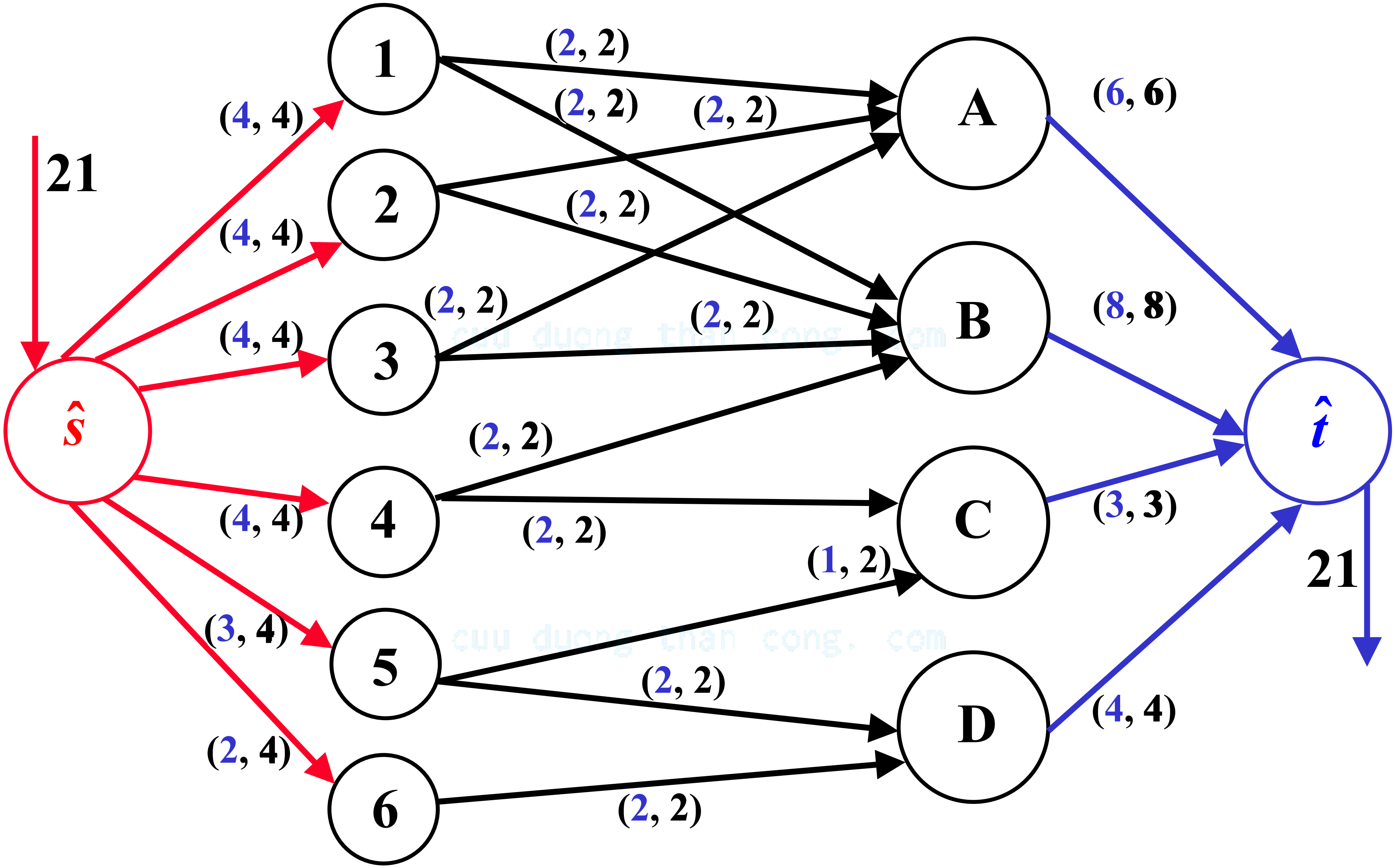
# APPLICATION TO MANPOWER SCHEDULING

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# SHORTEST ROUTE PROBLEM

- The problem is to determine the *shortest path* from $s$ to $t$ for a network with a set of nodes

$$\mathcal{N} = \{ s = 1, 2, \ldots , n = t \}$$

and arcs $(i, j)$, where for each arc $(i, j)$ of the network

$$d_{ij} = \text{distance or transit time}$$

- The determination of the shortest path from $1$ to $n$ requires the specification of the path

$$\{ (1, i_1), (i_1, i_2), \ldots, (i_q, n) \}$$

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# SHORTEST ROUTE PROBLEM

- We can write an $LP$ formulation of this problem in the form of a transshipment problem:

  ship $1$ unit from node $1$ to node $n$ by minimizing the shipping costs using the data parameters

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$$d_{ij} = \begin{cases} \textit{shipping costs for } 1 \textit{ unit from } i \textit{ to } j \\ \infty \textit{ whenever } i \textit{ and } j \textit{ are not directly connected} \end{cases}$$

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# THE DIJKSTRA ALGORITHM

- The solution is very efficiently performed using *the Dijkstra algorithm*
- The assumptions are
  - $d_{ij}$ is given for each pair of nodes
  - $d_{ij} \geq 0$
- The scheme is, basically, a label assignment procedure, which assigns nodes with either a *permanent* or a *temporary* label

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# THE DIJKSTRA ALGORITHM

- The *temporary* label of a node $i$ is an upper bound on the shortest distance from node $1$ to node $i$
- The *permanent* label is the actual shortest distance from node $1$ to node $i$

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# THE DIJKSTRA ALGORITHM

Step *0* : assign the *permanent* label *0* to node 1

Step 1 : assign *temporary* labels to all the other nodes

- $d_{1j}$ if node $j$ is directly connected to node 1
- ∞ if node $j$ is not directly connected to node 1

and select the minimum of the *temporary* labels and declare it *permanent* ; in case of ties, the choice is arbitrary

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# THE DIJKSTRA ALGORITHM

**Step 2 : let $\ell$ be the node most recently assigned a *permanent* label and consider each node $j$ with a *temporary* label; recompute the label to be**

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$$\min \left\{ \begin{matrix} \textit{temporary} \text{ label} \\ \text{of node } j \end{matrix} , \begin{matrix} \textit{permanent} \text{ label} \\ \text{of node } \ell \end{matrix} + d_{\ell j} \right\}$$

**Step 3 : select the smallest of the *temporary* labels and declare it *permanent* ; in case of ties, the choice is arbitrary**

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**Step 4 : if this is node $n$ , stop; else, go to Step 2**

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# THE DIJKSTRA ALGORITHM

- The shortest path is obtained by retracing the sequence of nodes with permanent labels starting from $n$ back to the node $1$

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- The path is then given in the forward direction starting from node $1$ and ending at node $n$

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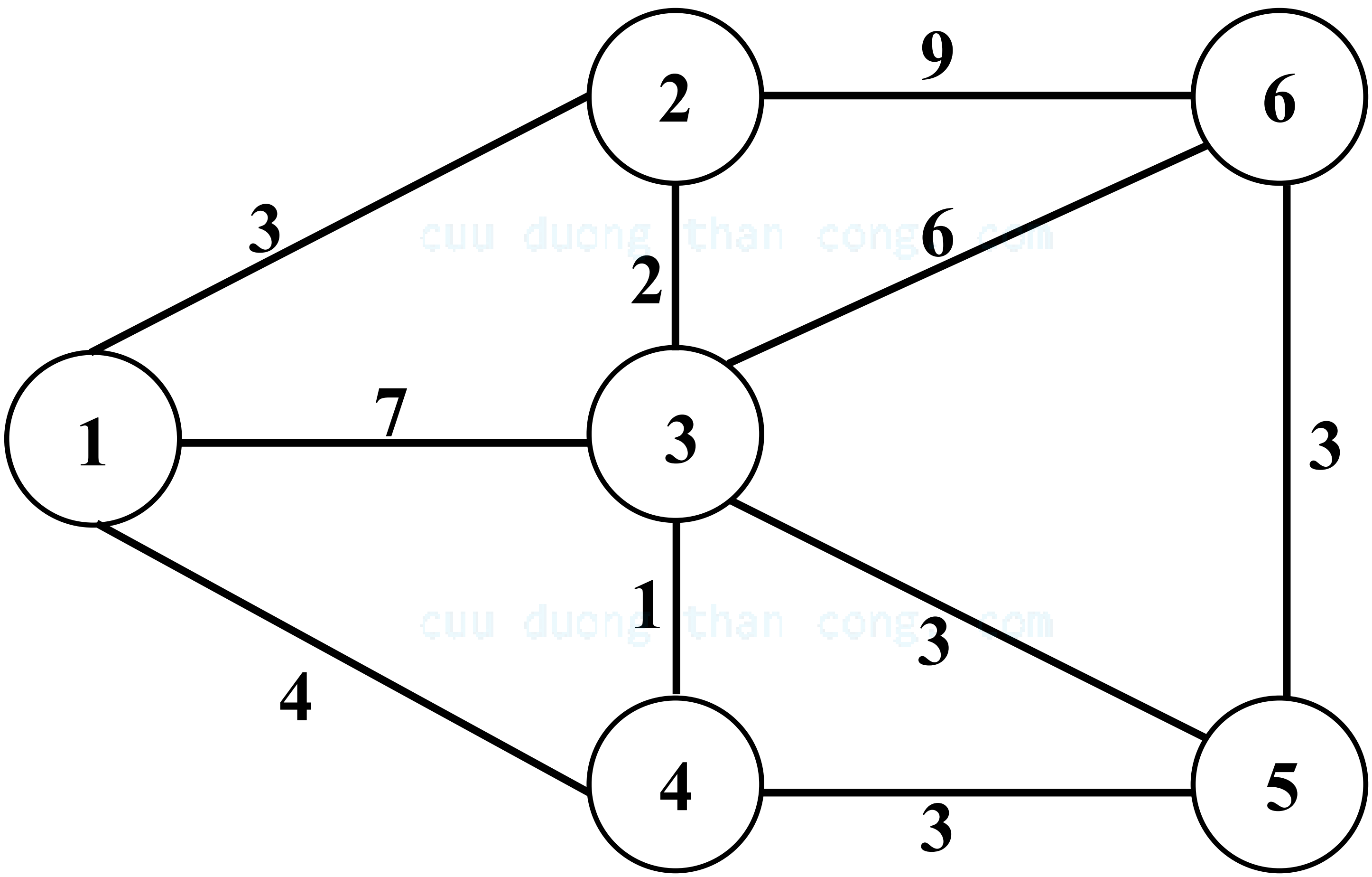
# EXAMPLE : SHORTEST PATH

- Consider the undirected network

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# EXAMPLE : SHORTEST PATH

- The problem is to
  - find the shortest path from 1 to 6
  - compute the length of the shortest path
- We apply the Dijkstra algorithm and assign iteratively a *permanent* label to each node

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# EXAMPLE : SHORTEST PATH

$$\boxed{0}$$

**Steps** *0* **and 1 :** $\mathscr{L}(0) = \left[ 0, 3, 7, 4, \infty, \infty \right]$

*initial label*

$$\boxed{1}$$

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**Steps 2 :** $\mathscr{L}(1) = \left[ 0, 3, 5, 4, \infty, 12 \right]$

*label in iteration 1*

$$\boxed{2}$$

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**Steps 2,3 and 4 :** $\mathscr{L}(2) = \left[ 0, 3, 5, 4, 7, 12 \right]$

*label in iteration 2*

$$\boxed{3}$$

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# EXAMPLE : SHORTEST PATH

**Steps 2,3 and 4** : $\mathscr{L}(3) = \left[0,3,5,4,7,11\right]$

$\boxed{4}$

*label in iteration 3*

**Steps 2,3 and 4** : $\mathscr{L}(4) = \left[0,3,5,4,7,10\right]$

$\boxed{5}$

*label in iteration 4*

$\mathscr{L}(4) = \left[0,3,5,4,7,10\right]$

$\boxed{6}$

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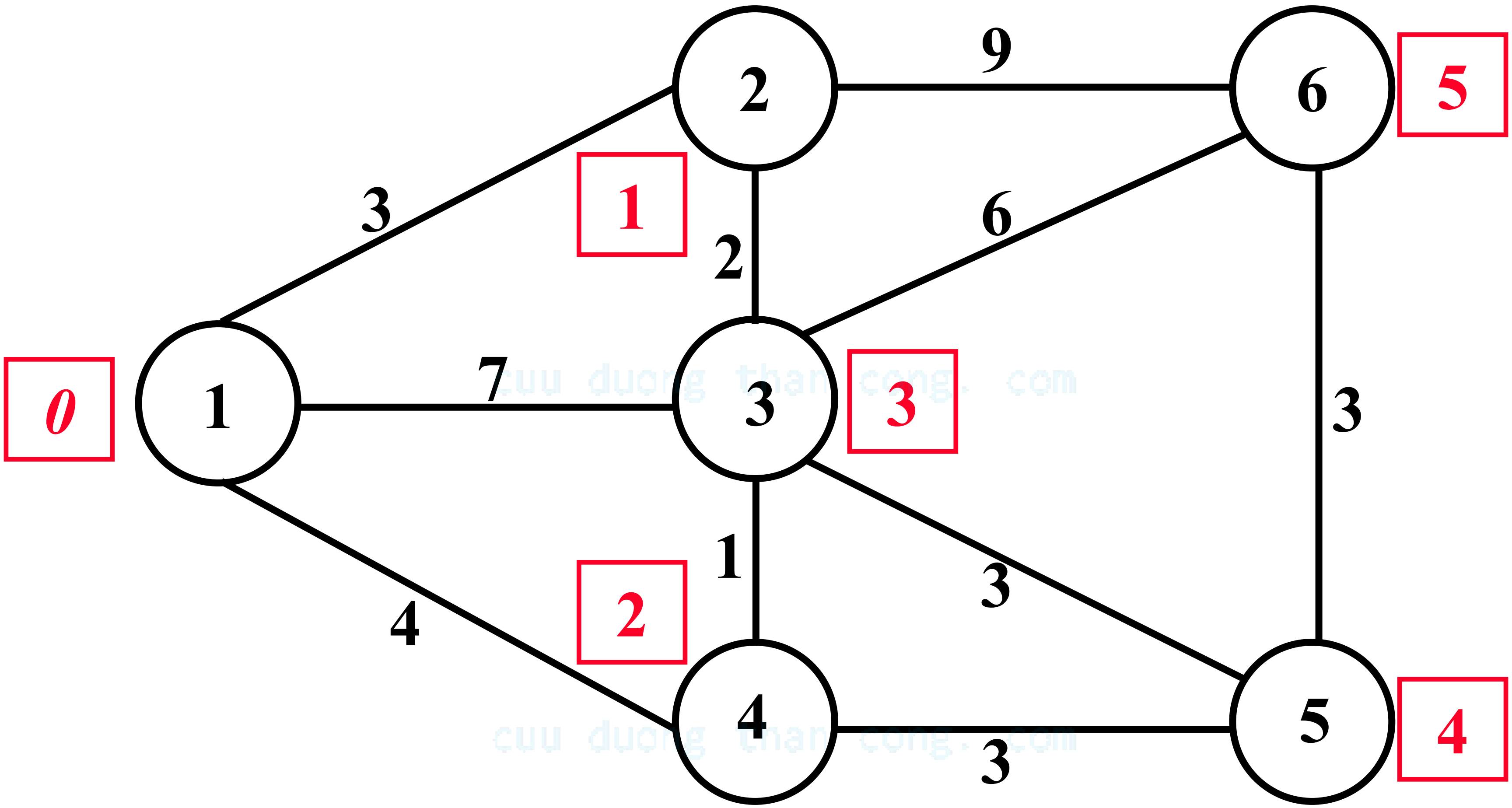
# EXAMPLE : SHORTEST PATH




- The shortest distance is 10 obtained from the path

{ ( 1, 4 ) , ( 4, 5 ) , ( 5, 6 ) }

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# PATH RETRACING

- We retrace the path from $n$ back to $1$ using the scheme:

  pick node $j$ preceding node $n$ as the node with the property

$$\textit{permanent label} \text{ of node } j \;+\; d_{jn} \;=\; \textit{shortest distance}$$

- In the retracing scheme, certain nodes may be skipped

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# SHORTEST PATH BETWEEN ANY TWO NODES

- The Dijkstra algorithm may be applied to determine the shortest distance between *any* pair of nodes $i$ , $j$ by taking $i$ as the *source* node and $j$ as the *sink* node

- We give as an example the following five – node network

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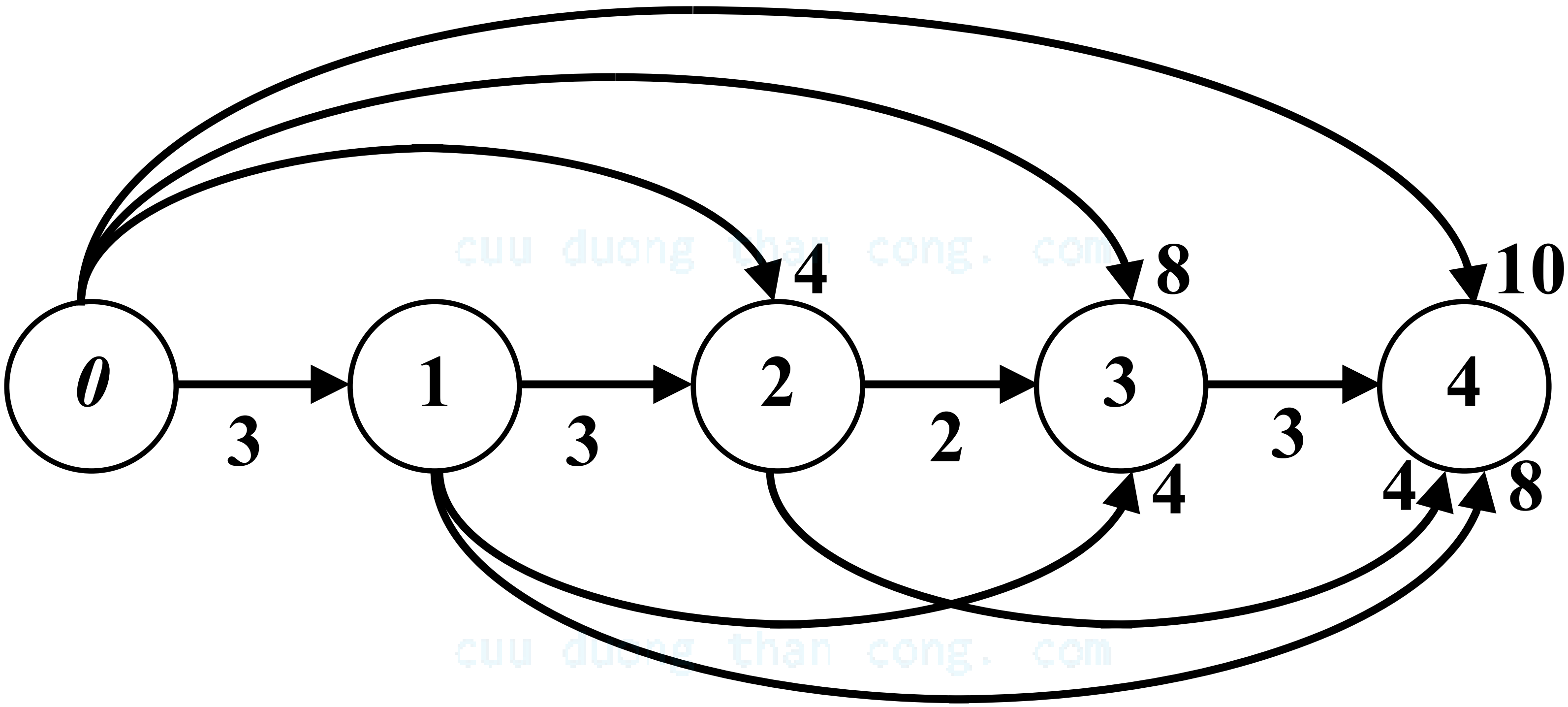
# EXAMPLE : FIVE – NODE NETWORK

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# EXAMPLE : FIVE – NODE NETWORK

$$\mathcal{L}(0) = \left[ 0,3,4,8,10,\infty \right]$$

$$\boxed{0}$$

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$$\mathcal{L}(1) = \left[ 0,3,4,7,10 \right]$$

$$\boxed{1}$$

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$$\mathcal{L}(2) = \left[ 0,3,4,6,8 \right]$$

$$\boxed{2}$$

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# EXAMPLE : FIVE – NODE NETWORK

$$\mathscr{L}(3) = \left[ 0, 3, 4, 6, 8 \right]$$

$$\boxed{3}$$

determines the shortest distance from $0$ to every other node

We retrace the path to get

$$8 = \underbrace{4}_{node\ 2} + \underbrace{d_{24}}_{4}$$

and so the path is

$$0 \rightarrow 2 \rightarrow 4$$

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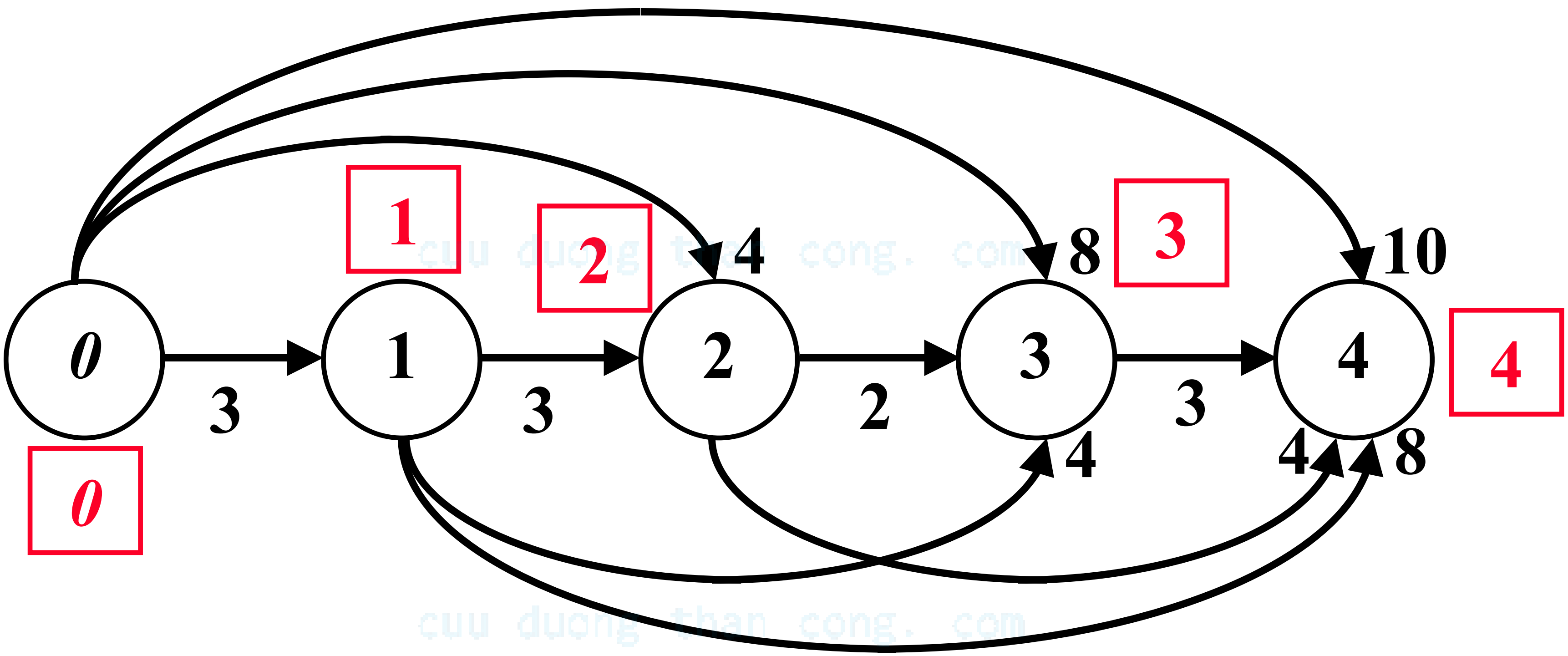
# EXAMPLE : FIVE – NODE NETWORK

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# APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

- We consider the problem of replacing old equipment or continuing its maintenance
- As equipment ages, the level of maintenance required increases and typically, this results in increased operating costs
- O&M costs may be reduced by replacing aging equipment; however, replacement requires addit-ional capital investment and so higher fixed costs

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# APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

- The problem is how often to replace equipment

  so as to minimize the total costs given by

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| | | | | |
|---|---|---|---|---|
| *total* | | *capital* | | *O&M* |
| | = | | + | |
| *costs* | | *costs* | | *costs* |
| | | ↑ | | ↑ |
| | | *fixed* | | *variable* |

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# EXAMPLE: EQUIPMENT REPLACEMENT

- Equipment replacement is planned during the next $5$ years
- The cost elements are

$p_j$ = *purchase costs in year* $j$

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$s_j$ = *salvage value of original equipment after* $j$ *years of use*

$c_j$ = *O&M costs in year* $j$ *of operation of equipment with the property that*

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$$\ldots \; c_j < c_{j+1} < c_{j+2} < \; \ldots$$

- We formulate this problem as a shortest route problem on a directed network

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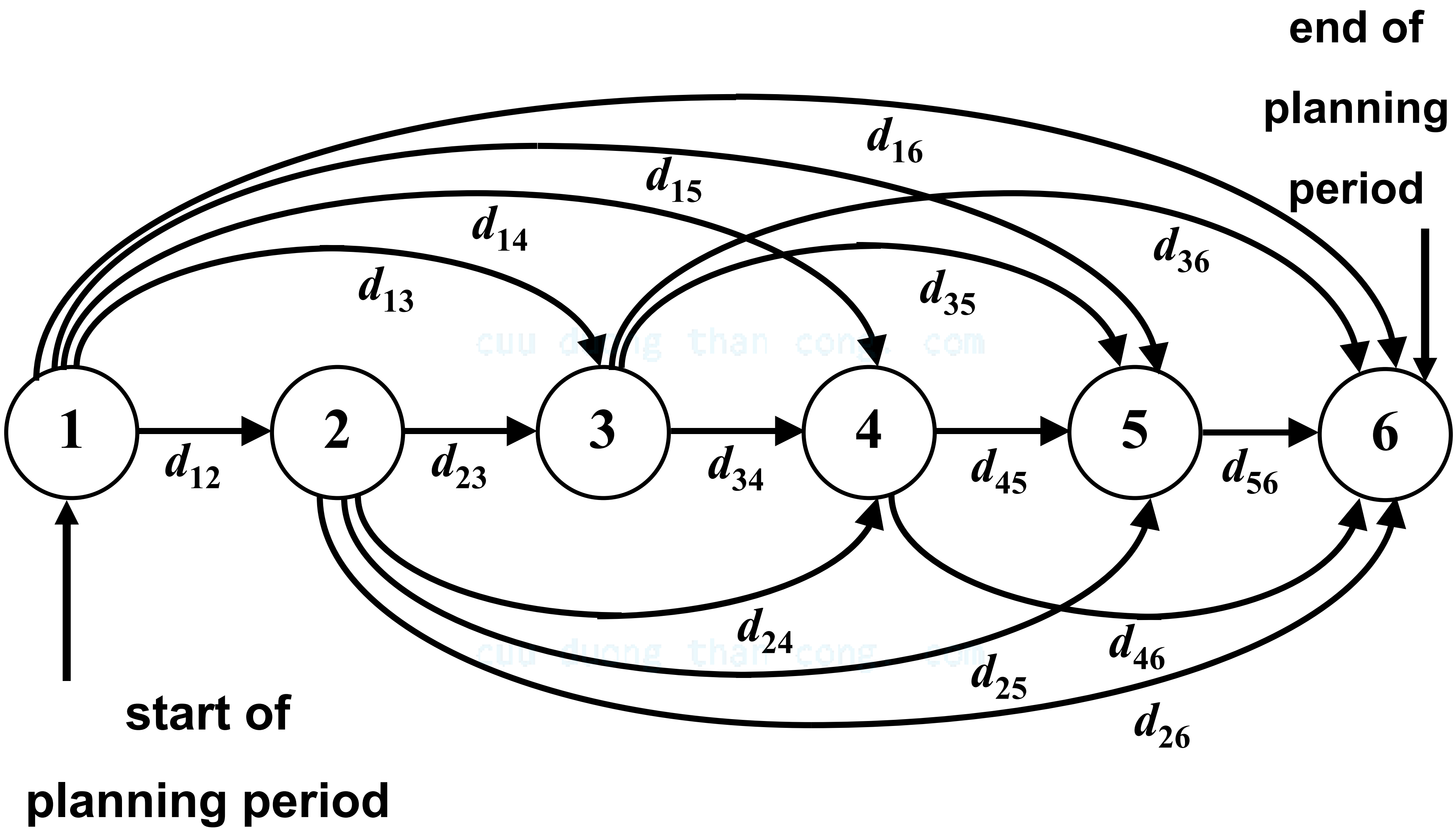
# EQUIPMENT REPLACEMENT PROBLEM

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# APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

where, the "distances" $d_{ij}$ are defined to be *finite* if $i < j$ , i.e., year $i$ precedes the year $j$ , with

$$d_{ij} = \underbrace{p_i}_{\substack{\textit{purchase}\\ \textit{price in}\\ \textit{year } i}} - \underbrace{s_{j-i}}_{\substack{\textit{salvage value}\\ \textit{after } j-i\\ \textit{years of use}}} + \underbrace{\sum_{\tau=1}^{j-i} c_\tau}_{\substack{\textit{O\&M costs}\\ \textit{for } j-i \textit{ years}\\ \textit{of operation}}} \qquad j > i$$

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# APPPLICATION : EQUIPMENT REPLACEMENT PROBLEM

- For example, if the purchase is made in year 1

$$d_{16} = p_1 - s_5 + \sum_{\tau=1}^{5} c_\tau$$

- The solution is the shortest distance path from year 1 to year 6 ; if for example the path is

$$\{ (1, 2), (2, 3), (3, 4), (4, 5), (5, 6) \}$$

then the solution is interpreted as the replacement of the equipment each year with

$$total\ costs = \sum_{i=1}^{5} p_i - 5s_1 + 5c_1$$

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# COMPACT BOOK STORAGE IN A LIBRARY

- This problem concerns the storage of books in a limited size library
- Books are stored according to their size, in terms of height and thickness, with books placed in groups of same or higher height; the set of book heights $\{ H_i \}$ is arranged in ascending order with

$$H_1 < H_2 < \ldots < H_n$$

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# COMPACT BOOK STORAGE IN A LIBRARY

- Any book of height $H_i$ may be shelved on a shelf of height at least $H_i$ , i.e., $H_i$ , $H_{i+1}$ , $H_{i+2}$ , . . .
- The length $L_i$ of shelving required for height $H_i$ is computed given the thickness of each book; the total shelf area required is $\sum_i H_i L_i$
  - if only $1$ height class [ corresponding to the tallest book ] exists, total shelf area required is the total length of the thickness of all books times the height of the tallest book

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# COMPACT BOOK STORAGE IN A LIBRARY

- if $2$ or more height classes are considered, the total area required is less than the total area required for a single class

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- The costs of construction of shelf areas for each height class $H_i$ have the components

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$s_i$ fixed costs [ independent of shelf area ]

$c_i$ variable costs / unit area

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# COMPACT BOOK STORAGE IN A LIBRARY

- For example, if we consider the problem with $2$ height classes $H_m$ and $H_n$ with $H_m < H_n$
  - all books of height $\leq H_m$ are shelved in shelf with the height $H_m$
  - all the other books are shelved on the shelf with height $H_n$
- The corresponding total costs are

$$\left[ s_m + c_m H_m \sum_{j=1}^{m} L_j \right] + \left[ s_n + c_n H_n \sum_{j=m+1}^{n} L_j \right]$$

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# COMPACT BOOK STORAGE IN A LIBRARY

- The problem is to find the set of shelf heights and lengths to *minimize* the *total shelving costs*
- The solution approach is to use a network flow model for a network with
  - the set of $(n+1)$ nodes

$$\mathcal{N} = \{\ 0, 1, 2, \ldots, n\ \}$$

corresponding to the $n$ book heights with

$$1 \leftrightarrow H_1 < H_2 < \ldots < H_n \leftrightarrow n$$

and the starting node with height 0

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# COMPACT BOOK STORAGE IN A LIBRARY

- **directed arcs** $(i, j)$ **only if** $j > i$ **resulting in**

  **a total of** $\dfrac{n(n+1)}{2}$ **arcs**

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- **"distance"** $d_{ij}$ **on each arc given by**

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$$d_{ij} = \begin{cases} s_j + c_j H_j \displaystyle\sum_{k=i+1}^{j} L_k & \text{if} \quad j > i \\ \infty & \text{otherwise} \end{cases}$$

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# COMPACT BOOK STORAGE IN A LIBRARY

- For this network, we solve the shortest route problem for the specified "distances" $d_{ij}$

- Suppose that for a problem with $n = 17$ , we determine the optimal trajectory to be

$$\{ (0,7), (7,9), (9,15), (15,17) \}$$

the interpretation of this solution is :

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# COMPACT BOOK STORAGE IN A LIBRARY

- store all the books of height $\leq H_7$ on the shelf of height $H_7$
- store all the books of height $\leq H_9$ but $> H_7$ on the shelf of height $H_9$
- store all the books of height $\leq H_{15}$ but $> H_9$ on the shelf of height $H_{15}$
- store all the books of height $\leq H_{17}$ but $> H_{15}$ on the shelf of height $H_{17}$

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