ECE 307 – Techniques for Engineering Decisions

Dynamic Programming

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DYNAMIC PROGRAMMING

- **Systematic approach to solving *sequential decision making* problems**

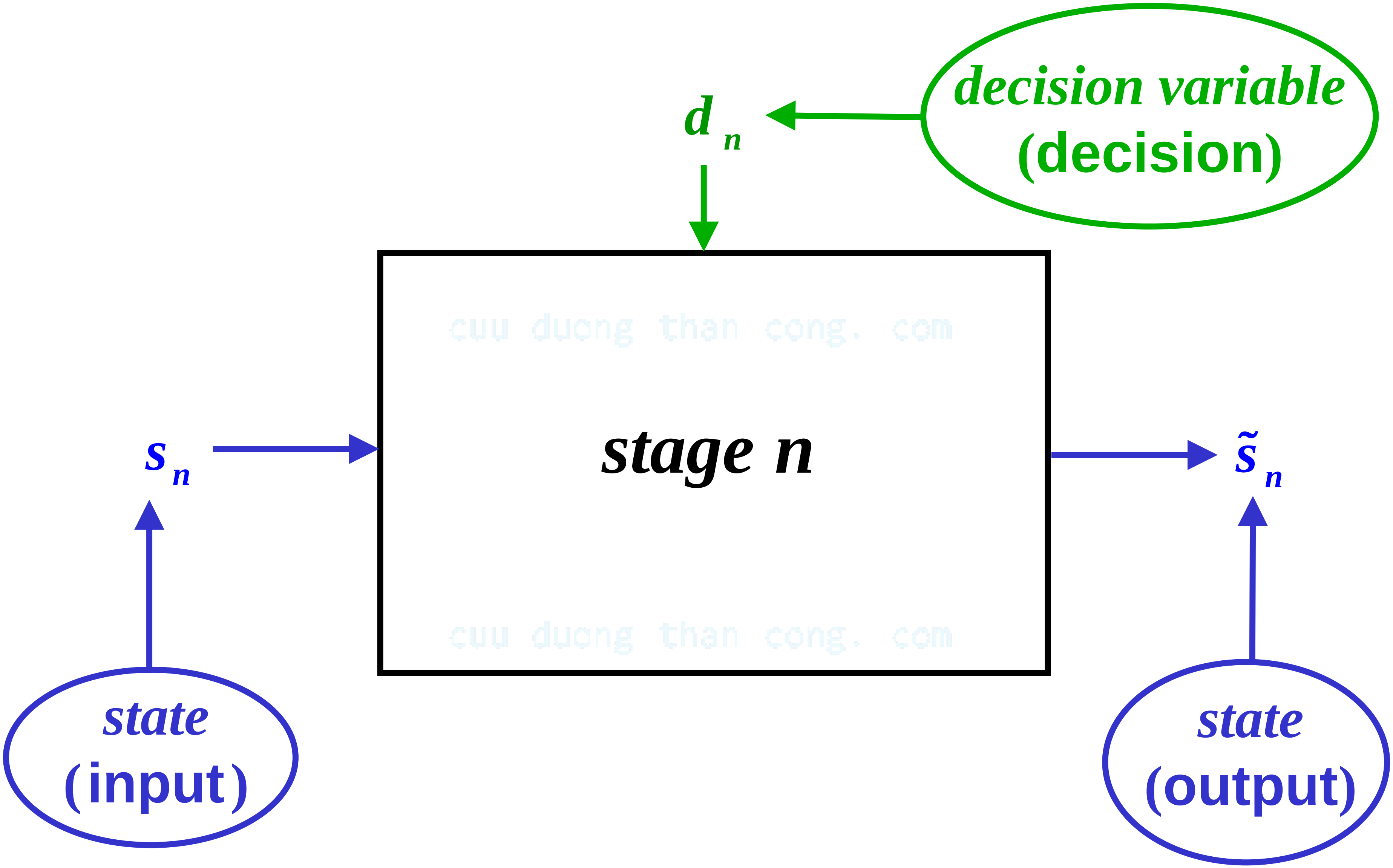
- **Salient problem characteristic: ability to *separate* the problem into *stages***

- ***Multi-stage* problem solving technique**

STAGES AND *STATES*

- We consider the problem to be composed of *multiple stages*
- A *stage* is the "point" in time, space, geographic location or structural element at which we make a decision; this "point" is associated with one or more *states*
- A *state* of the system describes a possible configuration of the system in a given *stage*

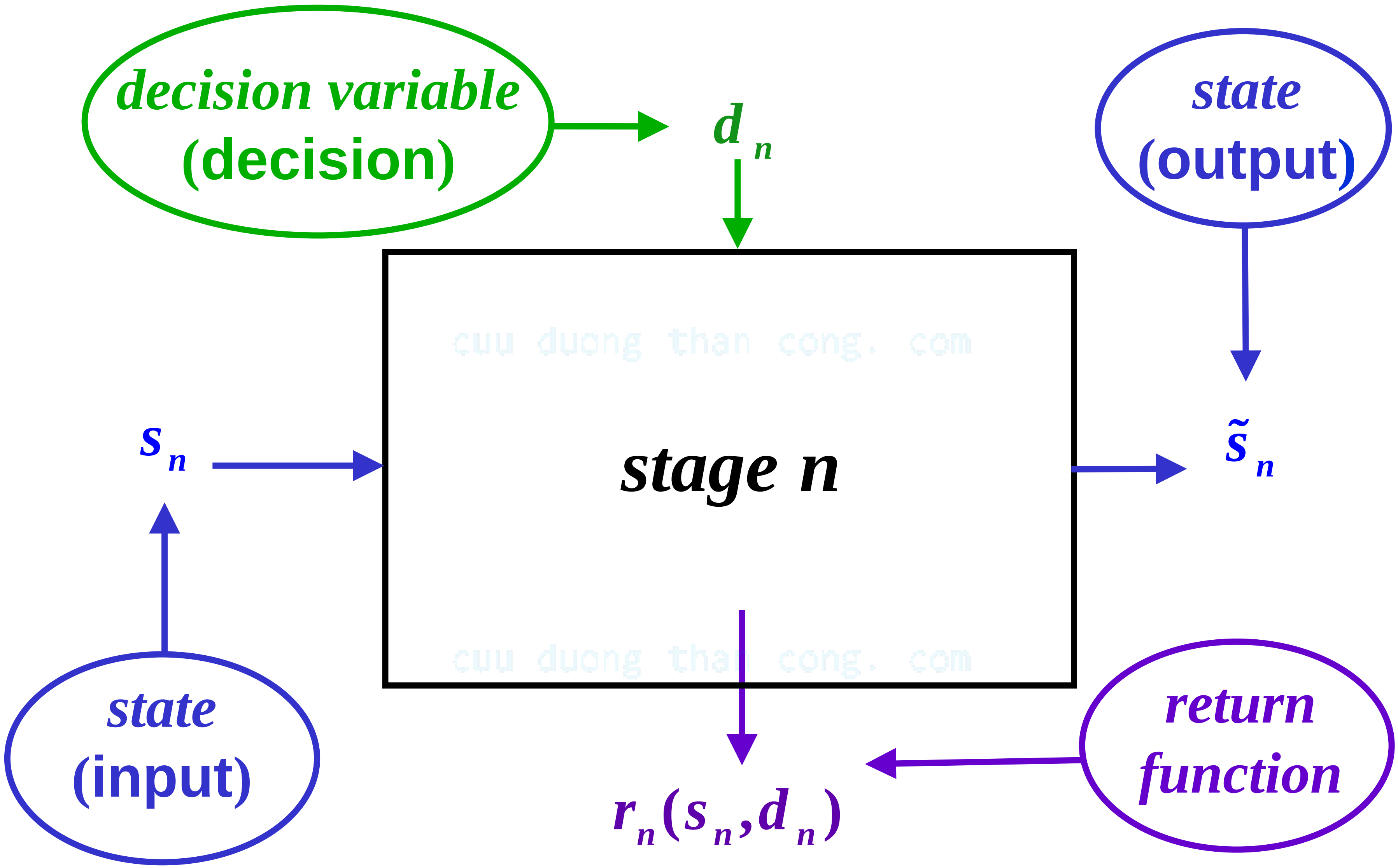
STAGES AND *STATES*



RETURN FUNCTION

- A *decision* d_n in the *stage n* *transforms* the *state* s_n in the *stage n* into the *state* s_{n+1} in the *stage* $n + 1$
- The *state* s_n and the *decision* d_n have an impact on the objective function; the effect is measured in terms of the *return function* denoted by $r_n(s_n,d_n)$
- The *optimal* decision at *stage* n is the *decision* d_n^* that optimizes the *return function* for the *state* s_n

RETURN FUNCTION



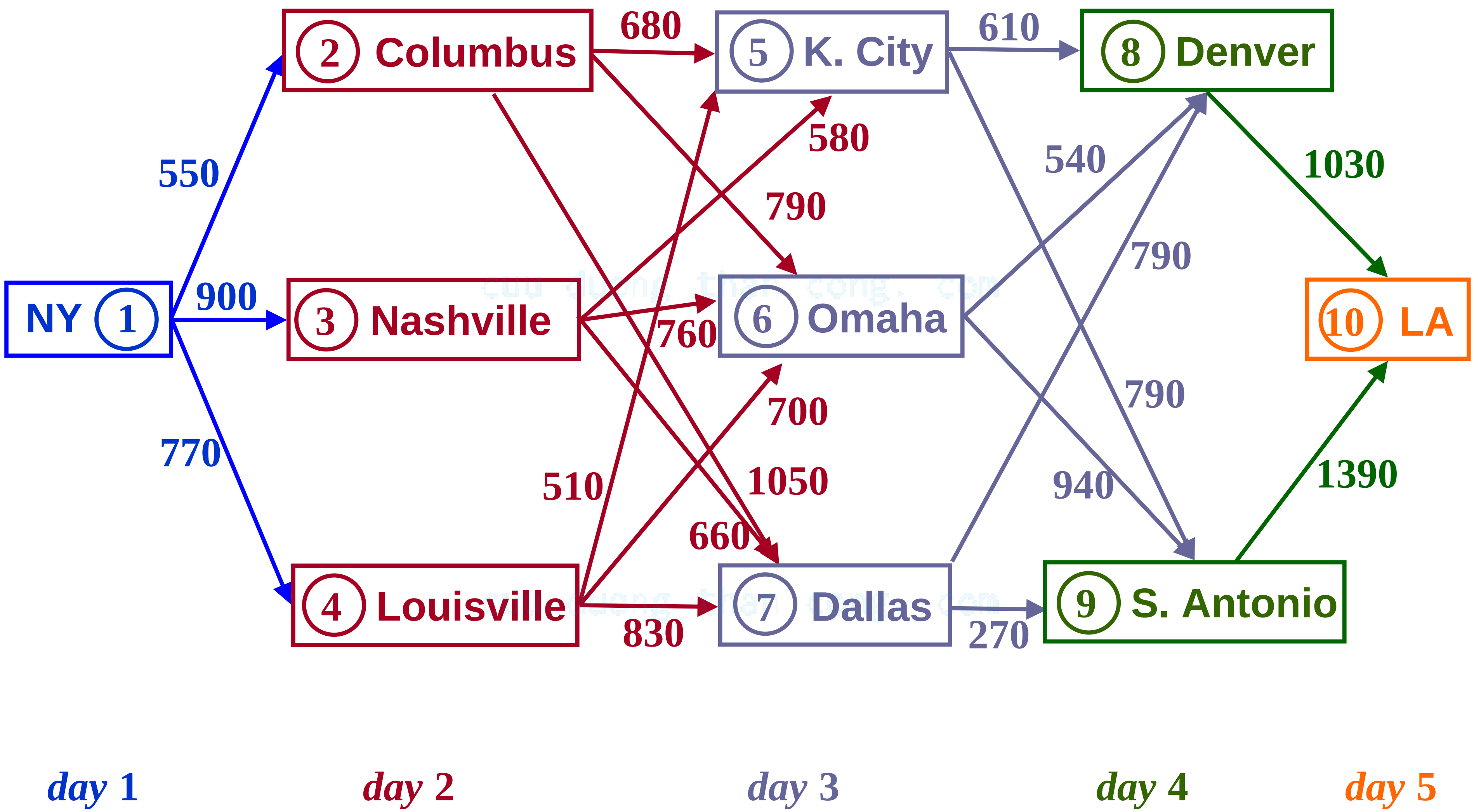
ROAD TRIP EXAMPLE

- A poor student is traveling from NY to LA
- To minimize costs, the student plans to sleep at friends' houses each night in cities along the trip

- Based on past experience he can reach
  - Columbus, Nashville or Louisville after 1 day
  - Kansas City, Omaha or Dallas after 2 days

  - San Antonio or Denver after 3 days
  - LA after 4 days

ROAD TRIP EXAMPLE



ROAD TRIP

- The student wishes to minimize the number of miles driven and so he wishes to determine the *shortest path* from NY to LA

- To solve the problem, he works *backwards*
- We adopt the following notation

c_{ij} = distance between *states* i and j

$f_k(i)$ = distance of the shortest path to LA from *state* i in the *stage* k

ROAD TRIP EXAMPLE CALCULATIONS

day 4 : $f_4(8) = 1{,}030 \qquad f_4(9) = 1{,}390$

day 3 : $$f_3(5) = \min\left\{ \underbrace{(610 + 1{,}030)}_{1{,}640}, \underbrace{(790 + 1{,}390)}_{2{,}180} \right\} = 1{,}640$$

$$f_3(6) = \min\left\{ \underbrace{(540 + 1{,}030)}_{1{,}570}, \underbrace{(940 + 1{,}390)}_{2{,}330} \right\} = 1{,}570$$

$$f_3(7) = \min\left\{ \underbrace{(790 + 1{,}030)}_{1{,}820}, \underbrace{(270 + 1{,}390)}_{1{,}660} \right\} = 1{,}660$$

ROAD TRIP EXAMPLE CALCULATIONS

$\boldsymbol{day}$ **2 :**

$$f_2(2) = min\left\{\underbrace{(680+1{,}640)}_{2{,}320},\underbrace{(790+1{,}570)}_{2{,}360},\underbrace{(1{,}050+1{,}660)}_{2{,}710}\right\} = 2{,}320$$

$$f_2(3) = min\left\{\underbrace{(580+1{,}640)}_{2{,}220},\underbrace{(760+1{,}570)}_{2{,}330},\underbrace{(660+1{,}660)}_{2{,}320}\right\} = 2{,}220$$

$$f_2(4) = min\left\{\underbrace{(510+1{,}640)}_{2{,}150},\underbrace{(700+1{,}570)}_{2{,}270},\underbrace{(830+1{,}660)}_{2{,}490}\right\} = 2{,}150$$

ROAD TRIP EXAMPLE

$day\ 1:$

$$f_1(1) = min\left\{\underbrace{(1,550+2,320)}_{*2,870*},\underbrace{(900+2,220)}_{3,120},\underbrace{(770+2,150)}_{2,920}\right\} = 2,870$$

- The shortest path is 2,870 miles and corresponds to the trajectory { (1, 2) , (2, 5) , (5, 8) , (8, 10) } , i.e., from NY, the student reaches Columbus on the first day, Kansas City on the second day, Denver the third day and then LA
- Every other trajectory to LA leads to higher costs and so is, by definition, *suboptimal*

PICK UP MATCHES GAME

- There are 30 matches on a table and 2 players
- Each player can pick up 1, 2, or 3 matches and continue until the last match is picked up
- The loser is the person who picks up the *last match*
- How can the player P_1, who goes first, ensure to be the winner?

WORKING BACKWARDS: PICK UP MATCHES GAME

- **We solve this problem by reasoning in a backwards fashion so as to ensure that when a single match remains, P_2 has the turn**
- **Consider the situation where 5 matches remain and it is P_2's turn; for P_1 to win we, consider all possible situations:**

WORKING BACKWARDS: PICK UP MATCHES GAME

P_2's move is to pick

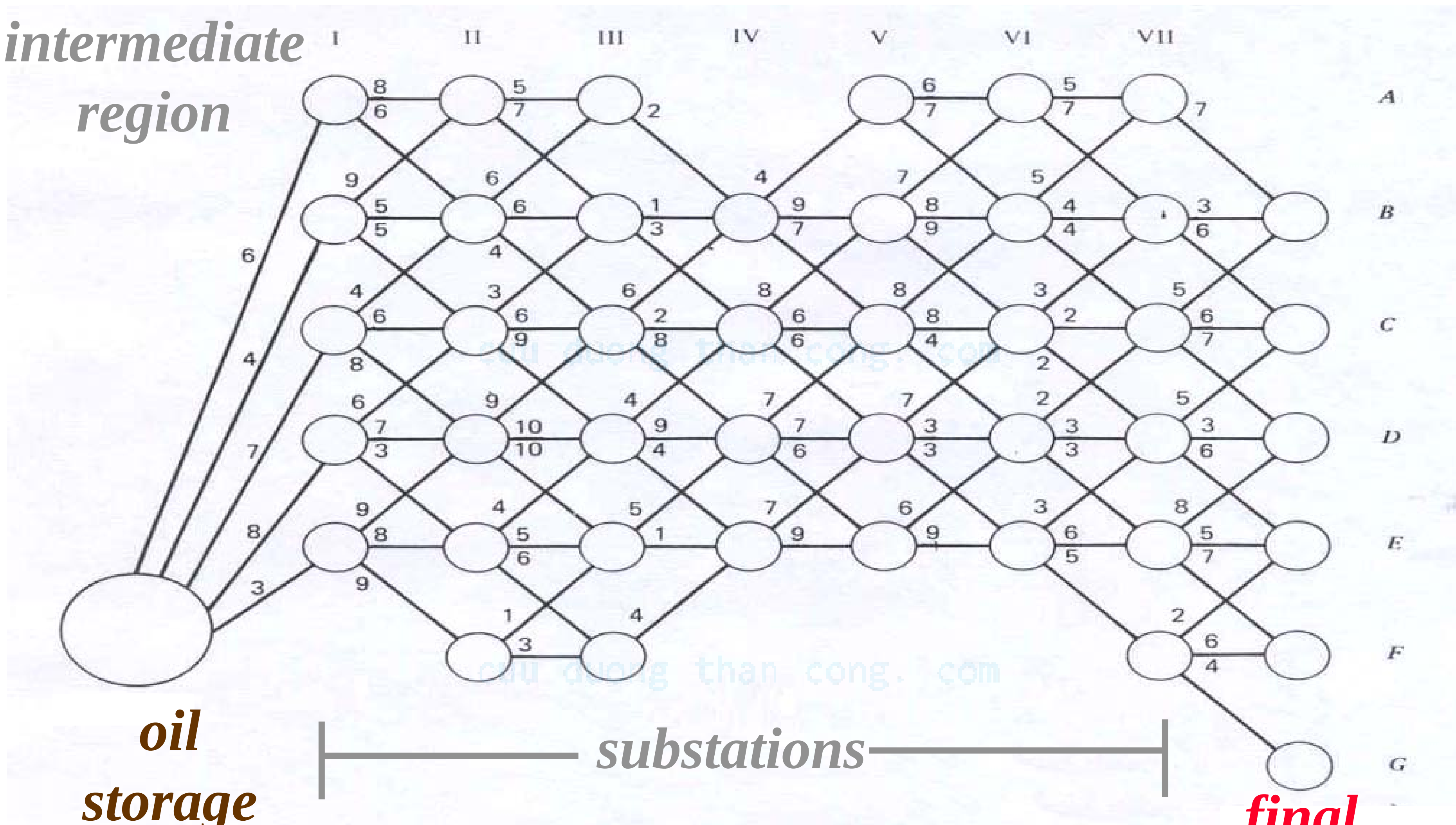
- $1 \Rightarrow$ 4 left → P_1 removes 3
- $2 \Rightarrow$ 3 left → P_1 removes 2
- $3 \Rightarrow$ 2 left → P_1 removes 1

- We can reason similarly for the cases of 9, 13, 17, 21, 25, and 29 matches
- Therefore, P_1 wins if P_1 picks 30 – 29 = 1 match in the first move
- In this manner, we can assure a win for any number of matches in the game

OIL TRANSPORT TECHNOLOGY

- We consider the development of a transport network from the north slope of Alaska to one of 6 possible shipping points in the U.S
- The network must meet the problem feasibility requirements
  - 7 pumping stations from a north slope ground storage plant to a shipping port
  - use of only those paths that are physically and environmentally feasible

OIL TRANSPORT TECHNOLOGY



OIL TRANSPORT TECHNOLOGY

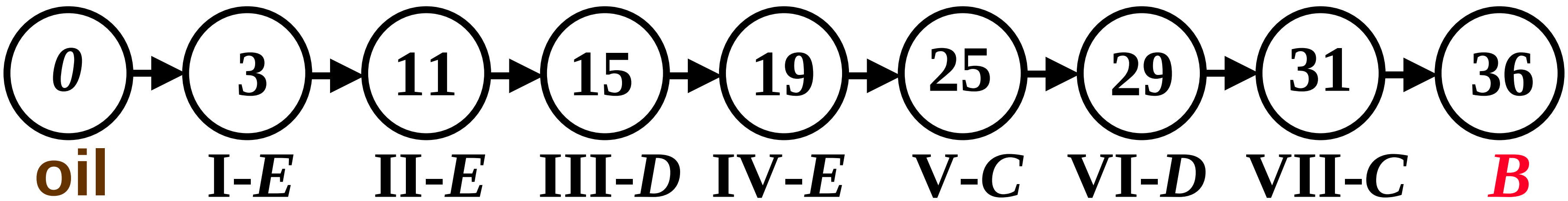
- **Objective: determine a *feasible* pumping configuration that minimizes the**

$$\textit{total costs} = \sum \textit{construction costs of the branches of an allowed path in the network of feasible pumping configurations}$$

OIL TRANSPORT TECHNOLOGY

- **Possible approaches to solving such a problem:**
  - ***enumeration*: exhaustive evaluation of all possible paths; too costly since there are more than 100 possible paths**
  - ***myopic decision rule*: at each node, pick as the next node the one reachable by the cheapest path (in case of ties the pick is arbitrary) ; for example,**

OIL TRANSPORT TECHNOLOGY



storage

but such a path is not unique and cannot be guaranteed to be *optimal*

- *serial dynamic programming* (*DP*) : we need to construct the problem solution by defining the *stages*, *states* and *decisions*

DP SOLUTION

- We define a *stage* to represent each pumping region and so each *stage* corresponds to the set of vertical nodes in the initial, the intermediate $I, II, \ldots, VII$ and the final regions
- We use *backwards recursion*: start from a final destination and work backwards to the oil storage *stage*

DP SOLUTION

- We define a ***state*** s_k to denote a final destination, a particular pumping station in the intermediate regions or the oil storage tank

- A decision refers to the selection of the branch from each ***state*** s_k, so there are at most three choices for a ***decision*** d_k:

$L \leftrightarrow$ left $\qquad F \leftrightarrow$ forward $\qquad R \leftrightarrow$ right

DP SOLUTION

- The *return function* $r_k(s_k, d_k)$ is defined as the costs associated with the *decision* d_k for the *state* s_k
- The transition function is the total costs in proceeding from a state in *stage* $k+1$ to another state in *stage* k , $k = 0, 1, \ldots, 7$
- We solve the problem by moving *backwards* iteratively starting from each final *state* to the *states* in the *stage* 1 and so on

DP SOLUTION: *STAGE* 1 ↔ REGION VII TO A FINAL DESTINATION

optimal decision → d_1^*

optimal return → $f_1^*(s_1)$

s_1	d_1			d_1^*	$f_1^*(s_1)$
	R	*L*	*F*		
A	7			*R*	7
B	6		3	*F*	3
C	7	5	6	*L*	5
D	6	5	3	*F*	3
E	7	8	5	*F*	5
F	4	2	6	*L*	2

least costs in proceeding from a state s_1 to a final destination

DP SOLUTION: *STAGE* 2 ↔ REGION VI TO *STAGE* 1

optimal decision

s_2	d_2			d_2^*	$f_2^*(s_2)$
	R	*L*	*F*		
A	10		12	*R*	10
B	9	12	7	*F*	7
C	5	6	7	*R*	5
D	8	7	6	*F*	6
E	7	6	11	*L*	6

cumulative costs in proceeding from a state s_2 to a final destination

STAGE 2 CALCULATION

costs of proceeding from the state s_2 to a state s_1 in stage 1

$$f_2^*(s_2) = \min_{d_2} \left(r_2(s_2, d_2) + \underbrace{f_1^*(s_1)} \right)$$

a function of only s_1

$$\Downarrow$$

for a given d_2, the state s_1 is set

DP SOLUTION: *STAGE* 3 ↔ REGION V TO *STAGE* 2

$$f_3^*(s_3) = \min_{d_3} \left\{ r_3(s_3, d_3) + f_2^*(s_2) \right\}$$

s_3	d_3			d_3^*	$f_3^*(s_3)$
	R	*L*	*F*		
A	14		16	*R*	14
B	14	17	15	*R*	14
C	10	5	13	*R*	10
D	9	12	9	*R, F*	9
E		12	15	*L*	12

cumulative costs in proceeding from a state s_3 to a final destination

DP SOLUTION:
STAGE 4 ↔ REGION IV TO *STAGE* 3

$$f_4^*(s_4) = \min_{d_4} \left\{ r_4(s_4, d_4) + f_3^*(s_3) \right\}$$

s_4	d_4			d_4^*	$f_4^*(s_4)$
	R	*L*	*F*		
B	17	18	23	*R*	17
C	15	22	16	*R*	15
D	18	17	16	*F*	16
E		16	21	*L*	16

cumulative costs in proceeding from a state s_4 to a final destination

DP SOLUTION: *STAGE* 5 ↔ REGION III TO *STAGE* 4

$$f_5^*(s_5) = \min_{d_5} \{ r_5(s_5, d_5) + f_4^*(s_4) \}$$

s_5	d_5			d_5^*	$f_5^*(s_5)$
	R	*L*	*F*		
A	19			*R*	19
B	18		18	*R,F*	18
C	24	23	17	*F*	17
D	20	19	25	*L*	19
E		21	17	*F*	17
F		20		*L*	20

cumulative costs in proceeding from a state s_5 to a final destination

DP SOLUTION:
STAGE 6 ↔ REGION II TO *STAGE* 6

$$f_6^*(s_6) = \min_{d_6}\left\{r_6(s_6, d_6) + f_5^*(s_5)\right\}$$

s_6	d_6			d_6^*	$f_6^*(s_6)$
	R	*L*	*F*		
A	25		24	*F*	24
B	21	25	24	*R*	21
C	28	21	23	*L*	21
D	27	26	29	*L*	26
E	26	23	22	*F*	22
F		18	23	*L*	18

cumulative costs in proceeding from a state s_6 to a final destination

DP SOLUTION: *STAGE* 7 ↔ REGION I TO *STAGE* 7

$$f_7^*(s_7) = \min_{d_7} \left\{ r_7(s_7, d_7) + f_6^*(s_6) \right\}$$

s_7	d_7			d_7^*	$f_7^*(s_7)$
	R	*L*	*F*		
A	27		32	*R*	27
B	26	33	26	*R,F*	26
C	34	25	27	*L*	25
D	25	27	33	*R*	25
E	27	35	30	*R*	27

cumulative costs in proceeding from a state s_7 to a final destination

THE *OPTIMAL* TRAJECTORY

- For the last *stage* corresponding the oil storage

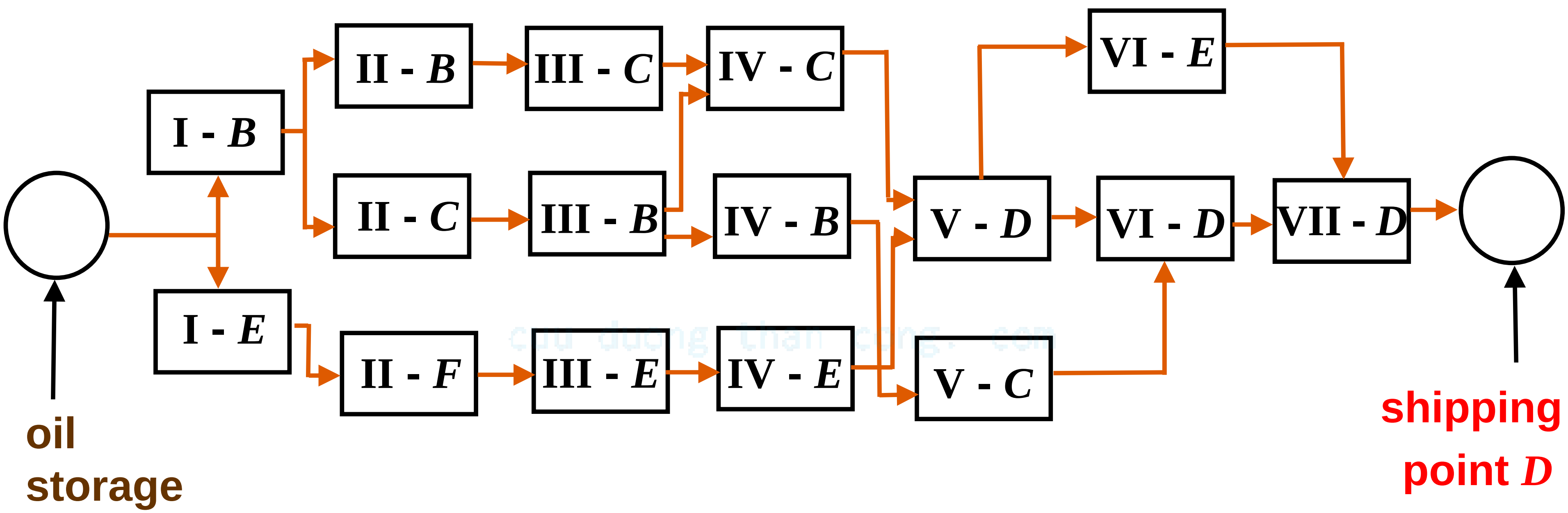
s_8 \ d_8	A	B	C	D	E	d_8^*	$f_8^*(s_8)$
$f_8(s_8)$	33	30	32	33	30	B,E	30

$$f_8^*(s_8) = min\{27+6,\ 26+4,\ 25+7,\ 25+8,\ 27+3\}$$

$$= 30$$

- To find the *optimal* trajectory, we retrace forwards proceeding through the *stages* 7, 6, . . . , 1 to get

THE *OPTIMAL* TRAJECTORY

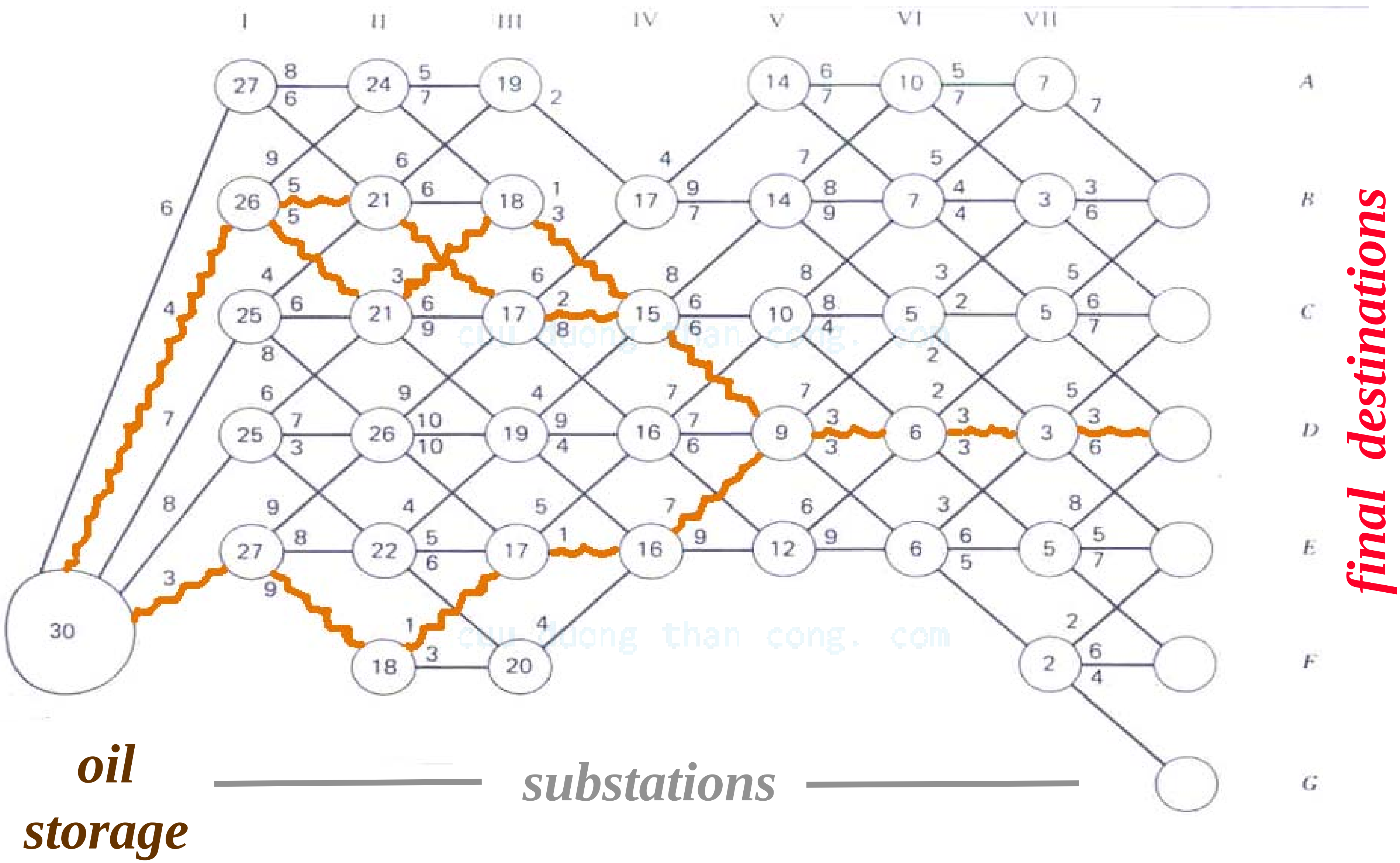


- **In addition to this *optimal* solution, other trajectories are possible since the path need not be unique but there is no path that yields a shorter total distance**

OIL TRANSPORT PROBLEM SOLUTION

- **We obtain the diagram shown on the next slide by retracing the steps of proceeding to a final destination at each *stage***
- **The solution**
  - **provides all the *optimal trajectories***
  - **is based on logically breaking up the problem into *stages* with the calculations in each *stage* being a function of the number of *states* in the *stage***
  - **provides also all the *suboptimal paths***

OIL TRANSPORT PROBLEM OPTIMAL SOLUTIONS



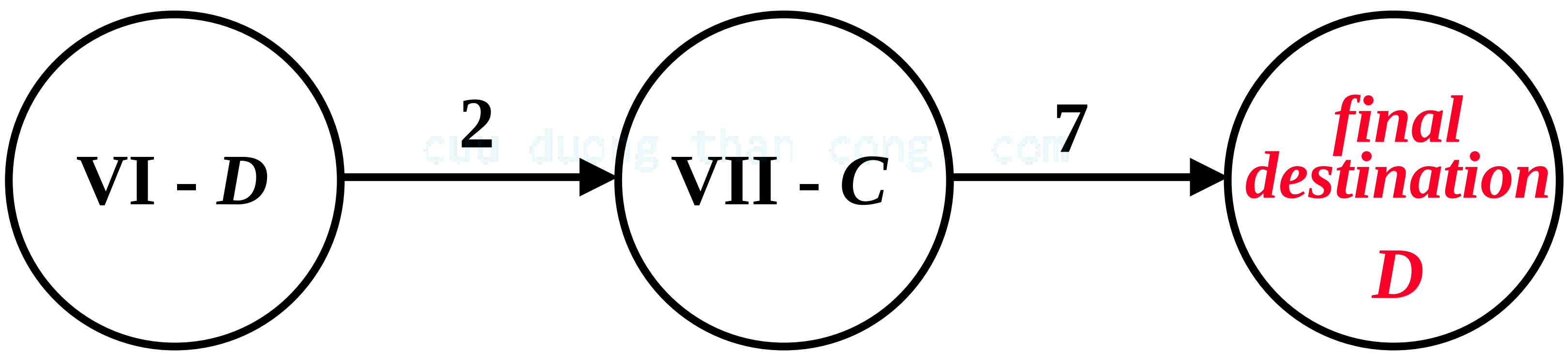
OIL TRANSPORT PROBLEM SOLUTION

- **For example, we may calculate the least cost *optimal* path to any *sub – optimal* shipping point different than *D***

- **From the solution, we can also determine the *sub–optimal* path if the construction of a feasible path is not undertaken**

OIL TRANSPORT: SENSITIVITY CASE

- Consider the case where we got to *stage* VI but the branch VI – *D* to VII – *D* cannot be built due to some environmental constraint
- We determine, then, the least-cost path from VI – D to find the final destination *D* whose value is 9 instead of 6



and so the sub *optimal* cost solution costs are 33

FACILITIES SELECTION PROBLEM

- A company is expanding to meet a wider market and considers:
  - 3 location alternatives
  - 4 different building types (sizes) at each site
- Revenues and costs vary with each location and building type

FACILITIES SELECTION PROBLEM

- Revenues R increase monotonically with building size; these are net revenues or profits

- Costs C increase monotonically with building size
- The data for building sizes and the associated revenues and costs are given in the table

FACILITIES SELECTION PROBLEM

site	building size									
	B_1		B_2		B_3		B_4		*none*	
	R_1	C_1	R_2	C_2	R_3	C_3	R_4	C_4	R_0	C_0
I	0.50	1	0.65	2	0.8	3	1.4	5	*0*	*0*
II	0.62	2	0.78	5	0.96	6	1.8	8	*0*	*0*
III	0.71	4	1.2	7	1.6	9	2	11	*0*	*0*

FACILITIES SELECTION PROBLEM

- The company can afford to invest at most 21 million *$* in the total expansion project

- The goal is to determine the *optimal* expansion policy, i.e., the buildings to be built at each site

DP SOLUTION APPROACH

- We use the *DP* approach to solve this problem; first, however, we need to define the *DP* structure elements

- For the facilities siting problem, we realize that without the choice of a site, the building type is irrelevant and so the elements that control the entire decision process are the building sites

DP SOLUTION APPROACH

stage	↔	site
state	↔	amount of funds available for construction
decision	↔	building type
return function	↔	revenues
transition function	↔	impact of a decision on the availablity of resources

DP SOLUTION APPROACH

- We use backwards *DP* to solve the problem and start with site I $\leftrightarrow$ *stage* 1 , a purely arbitrary choice, where this *stage* 1 represents the last decision in the 3 – *stage* sequence and so is made after the decision for the other two sites have been taken
- The amount of funds available is unknown since the decision at sites II and III are already made, and so

$$0 \leq s_1 \leq 21$$

DP SOLUTION APPROACH

- There are no additional decisions to be made in *stage 0* and we define

$$s_0 = 0 \quad \text{and} \quad f_0^*(s_0) = 0$$

- We start with *stage* 1 and move backwards to *stages* 2 and 3

- As we move backwards from *stage* $(n-1)$ to *stage* n, as a result of the *decision* d_n, the funds available for construction in *stage* $(n-1)$ are

DP SOLUTION APPROACH

$$s_{n-1} = s_n - c_n \leftarrow$$ costs of *decision* d_n

- The recursion relation is given by

$$f_n^*(s_n) = \max_{d_n} \left\{ f_n(s_n, d_n) + f_{n-1}^*(s_{n-1}) \right\}, \quad n = 1,2,3$$

with

$$s_{n-1} = s_n - c_n$$

and

$$f_n(s_n, d_n) = r_n(s_n, d_n) = R_n \leftarrow$$ revenues for *decision* d_n

DP SOLUTION: *STAGE* 1 ↔ SITE I

$$f_1^*(s_1) = \max_{0 \le d_1 \le 4} \underbrace{\{r_1(s_1,d_1)\}}_{R_1}$$

s_1 \ d_1	*0*	1	2	3	4	d_1^*	$f_1^*(s_1)$
$21 \ge s_1 \ge 5$	*0*	.50	.65	.80	1.40	4	1.40
$4 \ge s_1 \ge 3$	*0*	.50	.65	.80		3	.80
2	*0*	.50	.65			2	.65
1	*0*	.50				1	.50
0	*0*	*0*	*0*	*0*	*0*	*0*	*0*

DP SOLUTION: *STAGE* 2 ↔ SITE II

- The amount of funds s_2 available is unknown since the decision at site III is already made
- The value of d_2 is a function of s_2 and we construct a decision table using

$$f_2^*(s_2) = \max_{0 \le d_2 \le 4} \{ \underbrace{\underbrace{r_2(s_2, d_2)}_{R_2} + f_1^*(s_1)} \}$$

where

$$s_1 = s_2 - c_2$$

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DP SOLUTION: *STAGE* 2 ↔ SITE II

s_2 \ d_2	*0*	1	2	3	4	d^*_2	$f^*_2(s_2)$
$21 \geq s_2 \geq 13$	1.40	2.02	2.18	2.36	3.20	4	3.20
12	1.40	2.02	2.18	2.36	2.60	4	2.60
11	1.40	2.02	2.18	2.36	2.60	4	2.60
10	1.40	2.02	2.18	1.76	2.45	4	2.45
9	1.40	2.02	1.58	1.61	2.30	4	2.30
8	1.40	2.02	1.58	1.61	1.80	1	2.02
7	1.40	2.02	1.43	1.46		1	2.02
6	1.40	1.42	1.28	0.96		3	1.46
5	1.40	1.42	0.78			1	1.42
4	0.80	1.27				1	1.27
3	0.80	1.12				1	1.12
2	0.65	0.62				0	0.65
1	0.50					0	0.50
0	0.00					0	0.00

SAMPLE CALCULATIONS

- Consider the case $s_2 = 10$ and $d_2 = 0$; then,

$$C_2 = 0 \text{ and } R_2 = 0 ;$$

also therefore,

$$s_1 = 10 \text{ and } d_1^* = 4$$

so that

$$f_1^*(s_1) = 1.4;$$

consequently,

$$f_2(s_2) = 1.4$$

SAMPLE CALCULATIONS

- Consider next the case $s_2 = 10$ and $d_2 = 4$; then,

$$C_2 = 8 \text{ and } R_2 = 1.8 ;$$

also therefore,

$$s_1 = 2 \text{ and } d_1^* = 2$$

so that

$$f_1^*(s_1) = .65$$

consequently,

$$f_2(s_2) = 2.45$$

which we can show is the optimal value

$$f_2^*(s_2) = 2.45$$

DP SOLUTION : *STAGE* 3 ↔ SITE III

- At *stage* 3 , the first decision is actually taken and so exactly 21 million is available and $s_3 = 21$
- We compute the elements in the table using

$$f_3^*(s_3) = \max_{d_3} \{ \underbrace{r_3(s_3, d_3)}_{R_3} + f_2^*(s_2) \}$$

where

$$s_2 = s_3 - C_3$$

OPTIMAL SOLUTION

s_3	d_3					d_3^*	$f_3^*(s_3)$
	0	1	2	3	4		
21	3.20	3.91	4.40	4.20	4.45	4	4.45

- *Optimal* **profits are** 4.45 **million and the** *optimal* **path is obtained by retracing steps from** *stage* 3 **to** *stage* **1:**

OPTIMAL SOLUTION

$d_3^* = 4 \leftrightarrow$ construct B_4 at site III

$s_2 = s_3 - C_3 = 21 - 11 = 10$

$d_2^* = 4 \leftrightarrow$ construct B_4 at site II

$s_1 = s_2 - C_2 = 10 - 8 = 2$

$d_1^* = 2 \leftrightarrow$ construct B_2 at site I

$C_1 = 5$ and $C_1 + C_2 + C_3 = 21$

SENSITIVITY CASE

- We next consider the case where the maximum investment available is 15 million
- By inspection, the results in *stages* 1 and 2 remain unchanged; however, we must recompute *stage* 3 results with the 15 million limit

s_3	d_3					d_3^*	$f_3^*(s_3)$
	0	1	2	3	4		
15	3.2	3.31	3.22	3.06	3.27	1	3.31

SENSITIVITY CASE

- **The *optimal* solution obtains maximum profits of 3.31 million and the decision is as follows:**

$$d_3^* = 1 \leftrightarrow \text{construct } B_1 \text{ at site III}$$

$$s_2 = s_3 - C_3 = 15 - 4 = 11$$

$$d_2^* = 4 \leftrightarrow \text{construct } B_4 \text{ at site II}$$

$$s_1 = s_2 - C_2 = 11 - 8 = 3$$

$$d_1^* = 3 \leftrightarrow \text{construct } B_3 \text{ at site I}$$

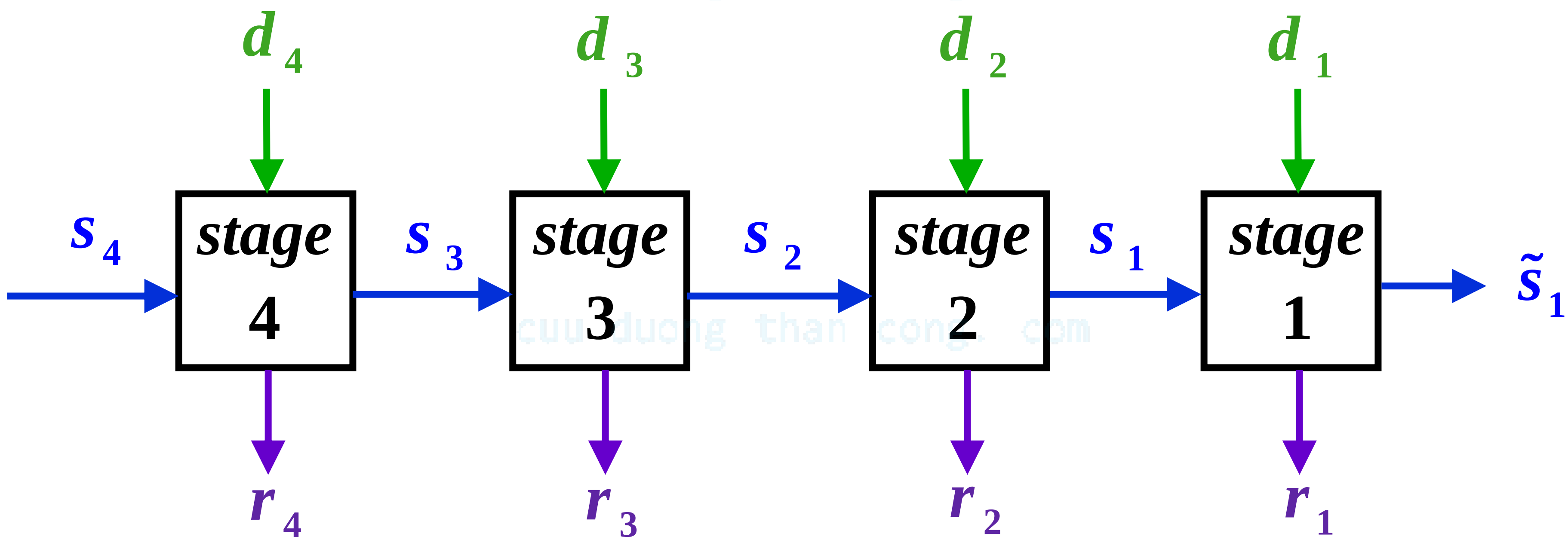
$$C_1 = 3 \text{ and } C_1 + C_2 + C_3 = 15$$

OPTIMAL CUTTING STOCK PROBLEM

- A paper company gets an order for:
  - 8 rolls of 2 *ft* paper at 2.50 *$/roll*
  - 6 rolls of 2.5 *ft* paper at 3.10 *$/roll*
  - 5 rolls of 4 *ft* paper at 5.25 *$/roll*
  - 4 rolls of 3 *ft* paper at 4.40 *$/roll*
- The company only has 13 *ft* of paper to fill these orders; partial orders can be filled
- Determine how to fill orders to maximize profits

DP SOLUTION APPROACH

- A *stage* is an order and since there are 4 orders we construct a $4-stage\ \ DP$



DP SOLUTION APPROACH

- **A *state* in *stage n* is the remaining *ft* of paper left for the order being processed at *stage n* and all the remaining *stages***
- **A decision in *stage n* is the amount of rolls to produce in *stage n* :**

DP SOLUTION APPROACH

$$d_n = \left\lfloor \frac{F_0}{L_n} \right\rfloor, \text{ the largest integer in } \frac{F_0}{L_n}$$

where

$$L_n = \text{length of order } n\ (ft)$$

$$F_0 = \text{length of available paper } (ft)$$

- The *return function* at *stage n* is the additional revenues gained from producing d_n *rolls*

DP SOLUTION APPROACH

- The *transition function* measures amount of paper remaining at *stage n*

$$s_{n-1} = s_n - d_n L_n \qquad n = 2,3,4$$

$$s_0 = s_1 - d_1 L_1$$

and s_0 should be as close as possible to *0*

- Clearly,

$$d_1 = \left\lfloor \frac{s_1}{L_1} \right\rfloor$$

DP SOLUTION APPROACH

- The recursion relation is

$$f_n^*(s_n) = \max_{0 \le d_n \le \left\lfloor \frac{s_n}{L_n} \right\rfloor} \left\{ r_n(s_n, d_n) + f_{n-1}^*(s_{n-1}) \right\}$$

where

$$s_{n-1} = s_n - d_n L_n$$

and

$$f_0^*(s_0) = 0$$

$$f_n(s_n, d_n) = r_n d_n + f_{n-1}^*(s_n - d_n L_n), \quad n = 1,2,3,4$$

DP SOLUTION APPROACH

- **We assume an arbitrary order of the *stages* and pick**

stage n	1	2	3	4
length of order (*ft*)	2.5	4	3	2

- **We proceed backwards from *stage* 1 to *stage* 4 and we know that**

DP SOLUTION: *STAGE* 1

$$f_1^*(s_1) = \max_{0 \le d_1 \le 5} \left\{ r_1(s_1, d_1) \right\} = \max_{0 \le d_1 \le 5} \{3.10\, d_1\}$$

$$d_1 \le \left\lfloor \frac{13}{2.5} \right\rfloor = 5$$

d_1 \ s_1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	—	—	—	3.10	3.10	→								
2	—	—	—	—	—	6.20	6.20	→						
3	—	—	—	—	—	—	—	—	9.30	9.30	→			
4	—	—	—	—	—	—	—	—	—	—	12.40	12.40	→	
5	—	—	—	—	—	—	—	—	—	—	—	—	—	15.50
$f_1^*(S_1)$	0	0	0	3.10	3.10	6.20	6.20	6.20	9.30	9.30	12.40	12.40	12.40	15.50
d^*_1	0	0	0	1	1	2	2	2	3	3	4	4	4	5

DP SOLUTION: *STAGE* 2

$$f_2^*(s_2) = \max_{0 \leq d_2 \leq 3} \left\{ 5.25\, d_2 + f_1^*(s_2 - 4\, d_2) \right\}$$

$$d_2 \leq \left\lfloor \frac{13}{4} \right\rfloor = 3$$

d_2 \ s_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	3.10	3.10	6.20	6.20	6.20	9.30	9.30	12.40	12.40	12.40	15.50
1	—	—	—	—	5.25	5.25	5.25	8.35	8.35	11.45	11.45	11.45	14.55	14.55
2	—	—	—	—	—	—	—	—	10.50	10.50	10.50	13.60	13.60	16.70
3	—	—	—	—	—	—	—	—	—	—	—	—	15.75	15.75
$f_2^*(s_2)$	0	0	0	3.10	5.25	6.20	6.20	8.35	10.50	11.45	12.40	13.60	15.75	16.70
d_2^*	0	0	0	0	1	0	0	1	2	1	0	2	3	2

DP SOLUTION: *STAGE* 3

$$f_3^*(s_3) = \max_{0 \le d_3 \le 4} \left\{ 4.40\, d_3 + f_2^*(s_3 - 3\, d_3) \right\}$$

$$d_3 \le \left\lfloor \frac{13}{3} \right\rfloor = 4$$

d_3 \ s_2	0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	0	3.10	5.25	6.20	6.20	8.35	10.50	11.45	12.40	13.60	15.75	16.70
1	—	—	—	4.40	4.40	4.40	7.50	9.65	10.60	10.60	12.75	14.90	15.85	16.80
2	—	—	—	—	—	—	8.80	8.80	8.80	11.90	14.05	15.0	15.0	17.15
3	—	—	—	—	—	—	—	—	—	13.20	13.20	13.20	16.30	18.45
4	—	—	—	—	—	—	—	—	—	—	—	—	17.60	17.60
$f_3^*(s_3)$	0	0	0	4.40	5.25	6.20	8.80	9.65	10.60	13.20	14.05	15.0	17.60	18.45
d_3^*	0	0	0	1	0	0	2	1	1	3	2	2	4	3

DP SOLUTION: *STAGE* 4

$$f_4^*(s_4) = \max_{0 \le d_4 \le 6} \left\{ 2.5\, d_4 + f_3^*(s_4 - 2\, d_4) \right\}$$

$$d_4 \le \left\lfloor \frac{13}{2} \right\rfloor = 6$$

d_4	0	1	2	3	4	5	6	d_4^*	$f_4^*(s_4)$
$s_4 = 13$	18.45	17.5	18.2	17.15	16.2	16.9	15	0	18.45

- The maximum profits are $\$18.45$

DP OPTIMAL SOLUTION

- The *optimal* solution is obtained by retracing

$$f_1^*(s_1 = 0) \;=\; 0 \quad \text{with } d_1^* = 0 \;\leftrightarrow\; \text{no rolls of } 2.5\ ft$$

$$f_2^*(s_2 = 4) \;=\; 5.25 \text{ with } d_2^* = 1 \;\leftrightarrow\; 1 \text{ roll of } 4\ ft$$

$$f_3^*(s_3 = 13) = 18.45 \text{ with } d_3^* = 3 \;\leftrightarrow\; 3 \text{ rolls of } 3\ ft$$

$$f_4^*(s_4 = 13) = 18.45 \text{ with } d_4^* = 0 \;\leftrightarrow\; \text{no rolls of } 2\ ft$$

SENSITIVITY CASE

- **Consider the case that due to an incorrect measurement, in truth, there are only 11 *ft* available for the rolls**

- **We note that the solution for the original 13 *ft* covers this possibility in the *stages* 1, 2 and 3 but we need to re-compute the results of *stage* 4, which we now call *stage* $4'$**

SENSITIVITY CASE : *STAGE* $4'$

- The *stage* $4'$ computations become

$$d_{4'} \leq \left\lfloor \frac{11}{2} \right\rfloor = 5$$

$d_{4'}$	*0*	1	2	3	4	5	$d_{4'}^*$	$f_{4'}^*(s_4)$
$s_4 = 11$	15	15.7	14.65	13.7	14.4	12.5	1	15.7

- The *optimal* profits in this sensitivity case are *$*15.7

SENSITIVITY CASE *OPTIMUM*

- The retrace of the solution path obtains

  - $d^*_{4'} = 1$ $\leftrightarrow$ 1 roll of 2 *ft*

  - $d^*_{3'} = 3$ $\leftrightarrow$ 3 rolls of 3 *ft*

  - $d^*_{2'} = 0$ $\leftrightarrow$ no rolls of 4 *ft*

  - $d^*_{1'} = 0$ $\leftrightarrow$ no rolls of 2.5 *ft*

ANOTHER SENSITIVITY CASE

- We consider the case with the initial 13 *ft*, but in addition we get the constraint that at least 1 roll of 2 *ft* must be produced:

$$d_4 \geq 1$$

- Note that no additional work is needed since the computations in the first tables have all the necessary data
- This sensitivity case *optimum* profits are $\$18.2$
- The *optimum* solution is :

OPTIMAL CUTTING STOCK PROBLEM

$f^*_{4''}(s_4 = 13)$ = 18.2 with $d^*_{4''} = 2$ $\leftrightarrow$ 2 rolls of 2 *ft*

$f^*_{3''}(s_3 = 9)$ = 13.2 with $d^*_{3''} = 3$ $\leftrightarrow$ 3 rolls of 3 *ft*

and since $s_2 = s_1 = 0$ $d^*_{2''} = 0$ $\leftrightarrow$ no rolls of 4 *ft*

$d^*_{1''} = 0$ $\leftrightarrow$ no rolls of 2.5 *ft*

- The constraint reduces *optimum* from *$* 18.45 to *$*18.2 and so it costs *$* .25

INVENTORY CONTROL PROBLEM

- **This problem is concerned with the development of an *optimal* ordering policy for a retailer**

- **The sales of a seasonal item has the demands**

month	*Oct*	*Nov*	*Dec*	*Jan*	*Feb*	*Mar*
demand	40	20	30	40	30	20

INVENTORY CONTROL PROBLEM

- **All units sold are purchased from a vendor at 4 *$/unit* ; units are sold in lots of 10, 20, 30, 40 or 50 with the corresponding discount**

lot size	10	20	30	40	50
discount %	4	5	10	20	25

INVENTORY CONTROL PROBLEM

- There are additional ordering costs: each order incurs fixed costs of $\$2$ and $\$8$ for shipping, handling and insurance
- The storage limitations of the retailer require that no more than 40 units be in inventory at the end of the month and the storage charges are $0.2\ \$/unit$; there is 0 inventory at the beginning and at the end of the period under consideration
- Underlying assumption: demand occurs at a constant rate throughout each month

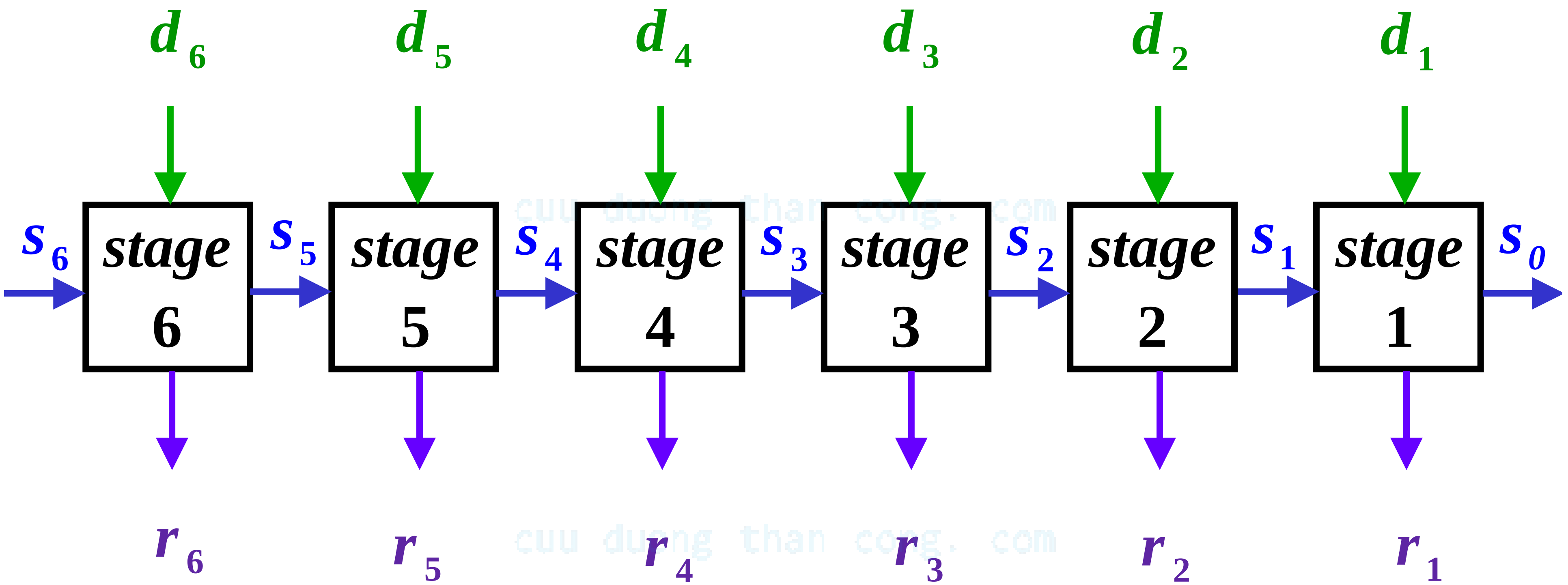
DP SOLUTION APPROACH

- We formulate the problem as a *DP* and use a backward process for solution

- Each *stage* corresponds to a month

month	*Oct*	*Nov*	*Dec*	*Jan*	*Feb*	*Mar*
stage *n*	6	5	4	3	2	1

DP SOLUTION APPROACH



DP SOLUTION APPROACH

- The *state* variable in *stage* n is defined as the amount of entering inventory given that there are n additional months remaining – the present month n plus the months $n-1, n-2, \ldots, 1$
- The decision variable d_n in *stage* n is the amount of units ordered to satisfy the demands D_i in the n remaining months, $i = 1, 2, \ldots, n$
- The transition function is defined by

$$s_{n-1} = s_n + d_n - D_n \qquad n = 1, 2, \ldots, 6$$

$$s_0 = 0 \qquad s_6 = 0$$

demand in month n

DP SOLUTION APPROACH

- The *return function* in the *stage* n is given by

$$r_1(s_1,d_1) = \underbrace{\phi(d_n)}_{\text{ordering costs}} + \underbrace{h_n(s_n + d_n - D_n)}_{0.2(s_n + d_n - D_n)\ \text{storage costs}}$$

with

$$d_n = 10, 20, 30, 40 \text{ or } 50$$

$$\phi(d_n) = \underbrace{10}_{\text{fixed costs}} + 4[1 - \underbrace{\rho(d_n)}_{\text{discount factor}}]\, d_n$$

DP SOLUTION APPROACH

d_n	0	10	20	30	40	50
$\phi(d_n)$	0	48	86	118	138	160

- In the *DP* approach, at each *stage* we minimize

$$f^*_n(s_n) = \min_{d_n}\left\{ \phi(d_n) + h_n\left[s_n + d_n - D_n\right] + f^*_{n-1}(s_{n-1})\right\}$$

$$n = 1, \dots, 6$$

$s_0 = 0$ and so $f^*_0(s_0) = 0$

DP SOLUTION: *STAGE* 1

$$\left.\begin{aligned} s_0 &= 0 \\ D_1 &= 20 \end{aligned}\right\} \Rightarrow s_1 = 20, 10 \text{ or } 0 \Rightarrow d_1^* = 0, 10 \text{ or } 20$$

$$f_1^*(s_1) = \min_{d_1} \left\{ \phi(d_1) + 0 \right\} = \phi(d_1^*)$$

s_1	20	10	*0*
d_1^*	*0*	10	20
$f_1^*(s_1)$	*0*	48	86

DP SOLUTION: *STAGE* 2

$$s_1 = s_2 + d_2 - 30 \text{ since } D_2 = 30$$

$$f_2^*(s_2) = \min_{d_2} \left\{ \phi(d_2) + 0.2\left[s_2 + d_2 - 30\right] + f_1^*(s_1) \right\}$$

s_2	d_2 = 0	10	20	30	40	50	d_2^*	$f_2^*(s_2)$
0				204	188	164	50	164
10			172	168	142		40	142
20		134	136	122	122		30	122
30	86	98	90				0	86
40	50	52					0	50

DP SOLUTION: *STAGE* 3

$$s_2 = s_3 + d_3 - 40 \text{ since } D_3 = 40$$

$$f_3^*(s_3) = \min_{d_3} \left\{ \phi(d_3) + 0.2\left[s_3 + d_3 - 40\right] + f_2^*(s_2) \right\}$$

s_3	d_3						d_3^*	$f_3^*(s_3)$
	0	10	20	30	40	50		
0					302	304	40	302
10				282	282	286	30, 40	282
20			250	262	264	252	20	250
30		212	230	244	230	218	10	218
40	164	192	212	210	196		*0*	164

DP SOLUTION: *STAGE* 4

$$s_3 = s_4 + d_4 - 30 \text{ since } D_4 = 30$$

$$f_4^*(s_4) = \min_{d_4} \left\{ \phi(d_4) + 0.2\left[s_4 + d_4 - 30 \right] + f_3^*(s_3) \right\}$$

s_4	d_4						d_4^*	$f_4^*(s_4)$
	0	10	20	30	40	50		
0				420	422	414	50	414
10			388	402	392	384	50	384
20		350	370	372	362	332	50	332
30	302	332	340	342	210		*0*	302
40	284	302	310	290			*0*	284

DP SOLUTION: *STAGE* 5

$$s_4 = s_5 + d_5 - 20 \text{ since } D_5 = 20$$

$$f_5^*(s_5) = \min_{d_5}\left\{ \phi(d_5) + 0.2\left[s_5 + d_5 - 20\right] + f_5^*(s_5) \right\}$$

s_5	d_5						d_5^*	$f_5^*(s_5)$
	0	10	20	30	40	50		
0			500	504	474	468	50	468
10		462	472	454	446	452	40	446
20	414	434	422	426	430		*0*	414
30	386	384	394	410			10	384
40	336	356	378				*0*	336

DP SOLUTION: *STAGE* 6

$D_6 = 40$ and $s_6 = 0$

$$s_5 = s_6 + d_6 - 40 = d_6 - 40$$

$$f_6^*(s_6) = \min_{d_6}\left\{ \phi(d_6) + 0.2\left[s_6 + d_6 - 40\right] + f_5^*(s_5) \right\}$$

d_6	0	10	20	30	40	50	d_6^*	$f_6^*(s_6)$
$f_6(s_6)$					606	608	40	606

$$d_6^* = 40 \Rightarrow d_5^* = 50 \Rightarrow d_4^* = 0 \Rightarrow d_3^* = 40 \Rightarrow d_2^* = 50 \Rightarrow d_1^* = 0$$

OPTIMAL SOLUTION

$d_6^* = 40$ which implies to $s_5 = 0$ and costs 606

$d_5^* = 50$ which implies to $s_4 = 30$ and costs 468

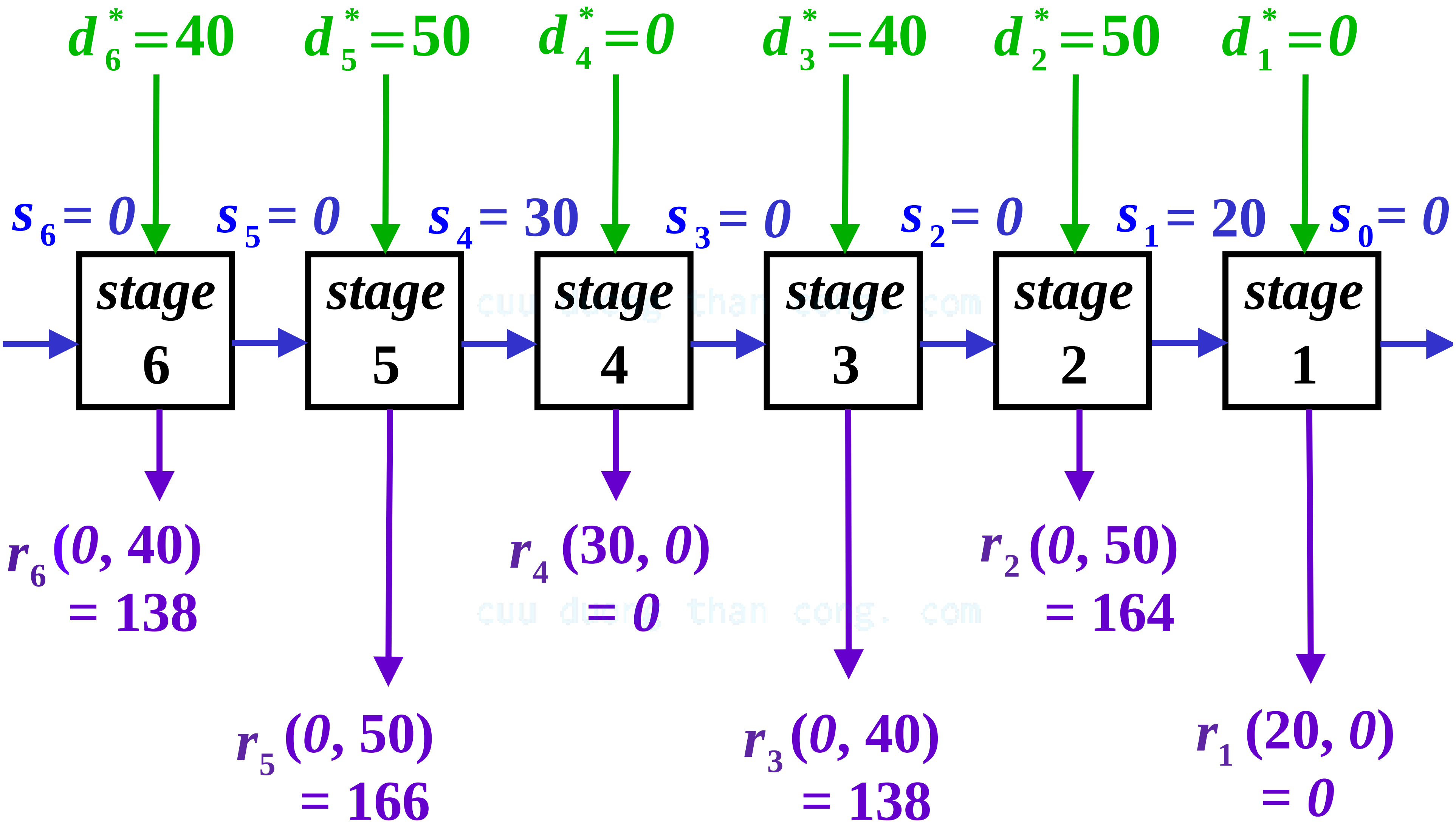
$d_4^* = 0$ which implies to $s_3 = 0$ and costs 302

$d_3^* = 40$ which implies to $s_2 = 0$ and costs 302

$d_2^* = 50$ which implies to $s_1 = 20$ and costs 164

$d_1^* = 0$ with costs *0*

OPTIMAL SOLUTION



OPTIMAL SOLUTION

optimal **trajectory is**

$$s_0 = 0 \rightarrow s_1 = 20 \rightarrow s_2 = 0 \rightarrow s_3 = 0 \rightarrow s_4 = 30 \rightarrow s_5 = 0$$

with total costs for the sequence of decisions of

$$0 + 164 + 138 + 0 + 166 + 138 = 606$$

MUTUAL FUND INVESTMENT STRATEGIES

- We consider a 5-year investment of
  - 10 *k$* invested in year 1
  - 1 *k$* invested in each year 2, 3, 4 and 5 into 2 mutual funds with different yields for both the short-term (1 year) and the long-term (up to 5 years)
- A decision at the beginning of each year is the allocation of investment in each fund

MUTUAL FUND INVESTMENT STRATEGIES

- We operate under the protocol that
  - once invested, the money cannot be withdrawn until the end of the 5 – year horizon
  - all short – term gains may be reinvested in either of the two funds or withdrawn in which case the withdrawn funds earn no further interest
- The objective is to maximize the total returns at the end of 5 years

MUTUAL FUND INVESTMENT STRATEGIES

- **The earnings on the investment are**
  - ***LTD*** **: the long-term dividend specified as** ***% / year*** **return on the accumulated capital**
  - ***STD*** **: the short-term interest dividend is the cash returned to the investor at the end of the period; cash may be reinvested and any money not invested in either of the funds earns nothing**

MUTUAL FUND INVESTMENT STRATEGIES

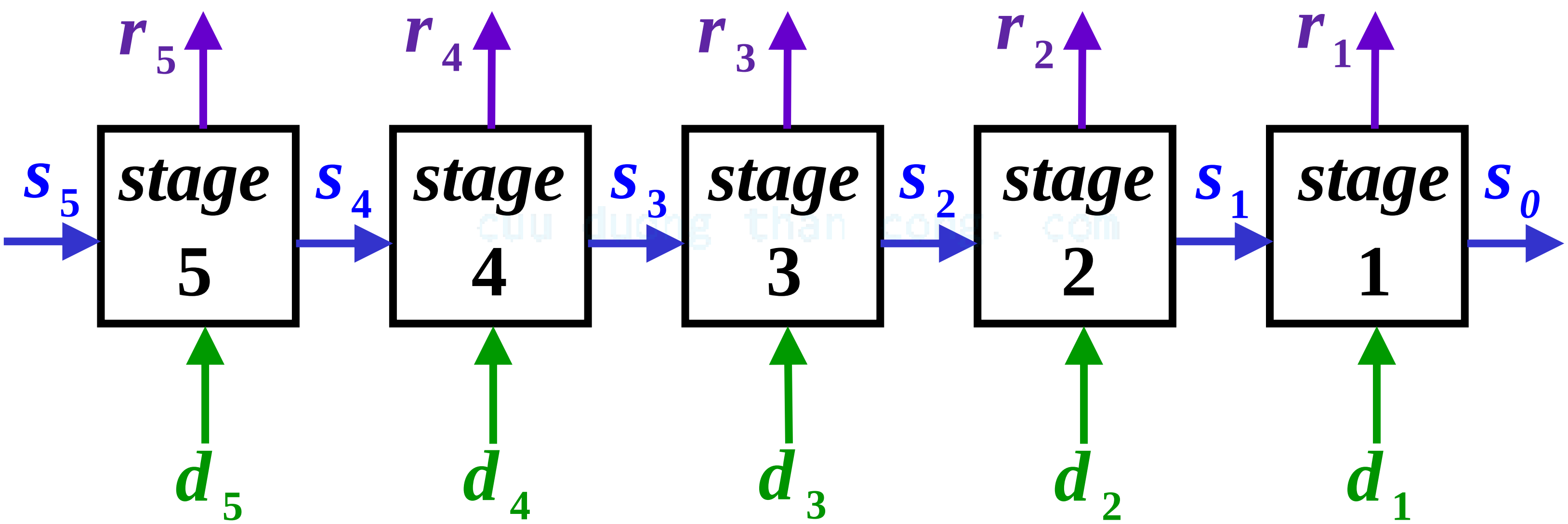
fund	*STD rate i_n for year n*					*LTD rate I*
	1	**2**	**3**	**4**	**5**	
A	**0.02**	**0.0225**	**0.0225**	**0.025**	**0.025**	**0.04**
B	**0.06**	**0.0475**	**0.05**	**0.04**	**0.04**	**0.03**

DP SOLUTION APPROACH

- We use backwards *DP* to solve the problem

- The *stages* are the 5 investment periods

$$stage\ n \triangleq \text{year } 6-n \qquad n = 1,2,3,4,5$$



DP SOLUTION METHOD

- For *stage* n , the *state* s_n is the amount of capital available for investment in the year $6-n$
- The decision d_n is the amount of capital invested in fund A in year $6-n$; the amount of capital invested in fund B in the year $6-n$ is therefore $s_n - d_n$
- In each year, we need to determine the amount to invest in fund A and in fund B

DP SOLUTION METHOD

- The use of *backward recursion* considers year 5 first and then each of the previous years in sequence
- Basic considerations:
  - for each year $6 - n, \; n = 1, 5$

    d_n is invested in fund A with returns $d_n i_A \, (SDT)$

    $(s_n - d_n)$ is invested in fund B with returns

    $(s_n - d_n) i_B \, (SDT)$

  - for the year $6 - n + 1$

$$s_{n-1} = d_n i_A + (s_n - d_n) i_B + 1000 \qquad n = 2,3,4,5$$

$$s_5 = 10{,}000$$

THE OBJECTIVE

- The objective is to maximize the total returns

$$max\ R = \sum_{n=1}^{5} r_n$$

- We express all returns in the end of the year 5 dollars: r_n is the future value of long – term earnings in the years 1, 2, 3 and 4

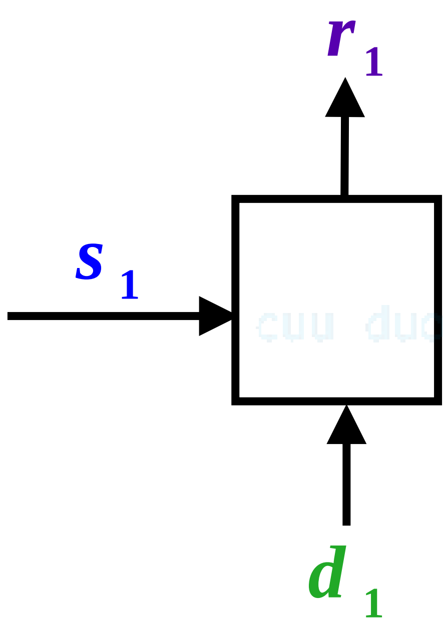
$$r_n = (1 + I_A)^n d_n + (1 + I_B)^n (s_n - d_n) \quad n = 1, \dots, 5$$

- But for $n = 1$, r_1 is the present value of all earnings in *stage* 1

$$r_1 = (1 + I_A)\, d_1 + (1 + I_B)\,(s_1 - d_1) + i_A d_1 + i_B (s_1 - d_1)$$

DP SOLUTION: *STAGE* 1

- For *stage* 1



where

$$r_1 = (1 + I_A) d_1 + (1 + I_B)(s_1 - d_1) + i_{1A} d_1 + i_{1B}(s_1 - d_1)$$

$$= \left(I_A + i_{1A} - I_B - i_{1B} \right) d_1 + (1 + I_B + i_{1B})\, s_1$$

DP SOLUTION: *STAGE* 1

- r_1 = earnings in *stage* 1 (returns realized at the end of 5 years)

$$f_1^*(s_1) = \max_{d_1} \{r_1\} = \max_{d_1} \left\{ \begin{array}{l} d_1 (I_A + i_{1A} - I_B - i_{1B}) + \\ s_1 (1 + I_B + i_{1B}) \end{array} \right\}$$

$$= \max_{0 \le d_1 \le s_1} \left\{ \begin{array}{l} d_1 (0.04 + 0.025 - 0.03 - 0.04) + \\ s_1 (1 + 0.03 + 0.04) \end{array} \right\}$$

$$= \max_{d_1} \left\{ d_1 (-0.005) + s_1 (1.07) \right\}$$

optimal decision → $d_1^* = 0$ **with** $f_1^*(s_1) = 1.07 s_1$ ← **maximum return in *stage* 1**

DP SOLUTION: *STAGE* 2

- r_2 = returns realized at the end of 5 years due to the decision in *stage* 2

$$= d_2(1+I_A)^2 + (s_2 - d_2)(1+I_B)^2$$

$$= d_2\left[(1+I_A)^2 - (1+I_B)^2\right] + s_2(1+I_B)^2$$

$$s_1 = s_2 i_{1B} + d_2(i_{1A} - i_{1B}) + 1{,}000$$

DP SOLUTION: *STAGE* 2

- We select d_2^* to maximize

$$f_2^*(s_2) = \max_{d_2} \left\{ r_2 + f_1^*(s_1) \right\}$$

$$= \max_{d_2} \left\{ d_2 (1.04^2 - 1.03^2) + s_2 (1.03)^2 + f_1^*(s_1) \right\}$$

$$= \max_{0 \le d_2 \le s_2} \left\{ \begin{array}{l} d_2 (.0207) + 1.0609 s_2 + \\ 1.07 [.04 s_2 + d_2 (-.015) + 1{,}000] \end{array} \right\}$$

$$= \max_{d_2} \left\{ d_2 (.0046) + 1.1037 s_2 + 1070 \right\}$$

$$d_2^* = s_2 \quad \text{with} \quad f_2^*(s_2) = 1.108 s_2 + 1070$$

DP SOLUTION: *STAGE* 3

- r_3 = returns realized at the end of 5 years due to the decision d_3

$$= d_3 (1 + I_A)^3 + (s_3 - d_3)(1 + I_B)^3$$

$$= d_3 \left[(1 + I_A)^3 - (1 + I_B)^3 \right] + s_3 (1 + I_B)^3$$

$$s_2 = s_3 \, i_{3B} + d_3 (i_{3A} - i_{3B}) + 1{,}000$$

DP SOLUTION: *STAGE* 3

- **We select d_3^* to maximize**

$$f_3^*(s_3) = \max_{d_3} \left\{ r_3 + f_2^*(s_2) \right\}$$

$$= \max_{d_3} \left\{ \begin{array}{l} d_3(1.04^3 - 1.03^3) + s_3(1.03)^3 + \\ 1.108 s_2 + 1,070 \end{array} \right\}$$

$$= \max_{0 \leq d_3 \leq s_3} \left\{ 2,178 + 1.1481 s_3 + .0018 d_3 \right\}$$

$$d_3^* = s_3 \quad \text{with} \quad f_3^*(s_3) = 1.15 s_3 + 2,178$$

DP SOLUTION: *STAGE* 4

- r_4 = returns realized at the end of 5 years due to the decision d_4

$$= d_4 (1 + I_A)^4 + (s_4 - d_4)(1 + I_B)^4$$

$$= d_4 \left[(1 + I_A)^4 - (1 + I_B)^4 \right] + s_4 (1 + I_B)^4$$

$$s_3 = s_4 \, i_{4B} + d_4 (i_{4A} - i_{4B}) + 1,000$$

DP SOLUTION: *STAGE* 4

- We select d_4^* to maximize

$$f_4^*(s_4) = \max_{d_4} \left\{ r_4 + f_3^*(s_3) \right\}$$

$$= \max_{d_4} \left\{ d_4 (1.04^4 - 1.03^4) + s_4 (1.03)^4 + 1.15 s_3 + 2{,}178 \right\}$$

$$= \max_{0 \le d_4 \le s_4} \left\{ 3328 + 1.1772 s_4 + .0156 d_4 \right\}$$

$d_4^* = s_4$ with $f_4^*(s_4) = 1.193 s_4 + 3{,}328$

DP SOLUTION: *STAGE* 5

- r_5 = returns realized at the end of 5 years due to the decision d_5

$$= d_5 (1 + I_A)^5 + (s_5 - d_5)(1 + I_B)^5$$

$$= d_5 \left[1.04^5 - 1.03^5 \right] + s_5 \ (1.03)^5$$

s_5 = $10,000 \leftarrow$ capital available for investment

$$s_4 = s_5 \, i_{5B} + d_5 (i_{5A} - i_{5B}) + 1,000$$

$$= 10,000 \, i_{5B} + d_5 (i_{5A} - i_{5B}) + 1,000$$

DP SOLUTION: *STAGE* 5

- We select d_5^* to maximize

$$f_5^*(s_5) = \max_{0 \le d_5 \le s_4} \left\{ \underbrace{10{,}000(1.03)^5}_{11{,}593} + d_5 \underbrace{(1.04^5 - 1.03^5)}_{0.0574} + f_4^*(s_4) \right\}$$

$$\left[1{,}000 + 600 + d_5(-.04)\right] 1.193 + 3{,}328$$

DP SOLUTION: *STAGE* 5

$$= \max_{0 \leq d_5 \leq s_5} \left\{ 16{,}830 + d_5 \frac{(.0574 - 0.048)}{0.097} \right\}$$

$$= 16{,}830 + 0.097(10{,}000)$$

$d_5^* = 10{,}000$ **with** $f_5^*(s_5) = 16{,}927$

OPTIMAL SOLUTION

optimal **return at end of 5 years is 16,927 using the following strategy**

beginning of year	investment in	
	fund *A*	fund *B*
1	10,000	*0*
2	*STD returns* + 1,000	*0*
3	*STD returns* + 1,000	*0*
4	*STD returns* + 1,000	*0*
5	*0*	*STD returns* + 1,000

