ECE 307 - Techniques for Engineering Decisions

Hungarian Method

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ASSIGNMENT PROBLEM

We are given

n machines $M_1, M_2, \ldots, M_n \leftrightarrow i$

n jobs
$$J_1, J_2, \dots, J_n \leftrightarrow j$$

 $c_{ij} = \begin{cases} \text{cost of doing job } j \text{ on machine } i \\ Q \text{ if job } j \text{ cannot be done on machine } i \end{cases}$

Each machine can only do one job and each job

requires one machine

ASSIGNMENT PROBLEM

□ We wish to determine the optimal match, i.e., the

assignment with the lowest total costs of doing

the jobs on the *n* machines

□ The brute force approach is simply enumeration:

consider n = 10 and there are 3,628,800 possible

choices!

SOLUTION APPROACH

We can, however, introduce *categorical* decision variables

$$x_{ij} = \begin{cases} 1 & \text{job } j \text{ is assigned to machine } i \\ 0 & \text{otherwise} \end{cases}$$

And the constraints can be stated as

 $\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \text{ each machine does exactly 1 job}$ $\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j \text{ each job is assigned}$

to only 1 machine

□ The assignment problem, then, is formulated as

min
$$Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

s.t.

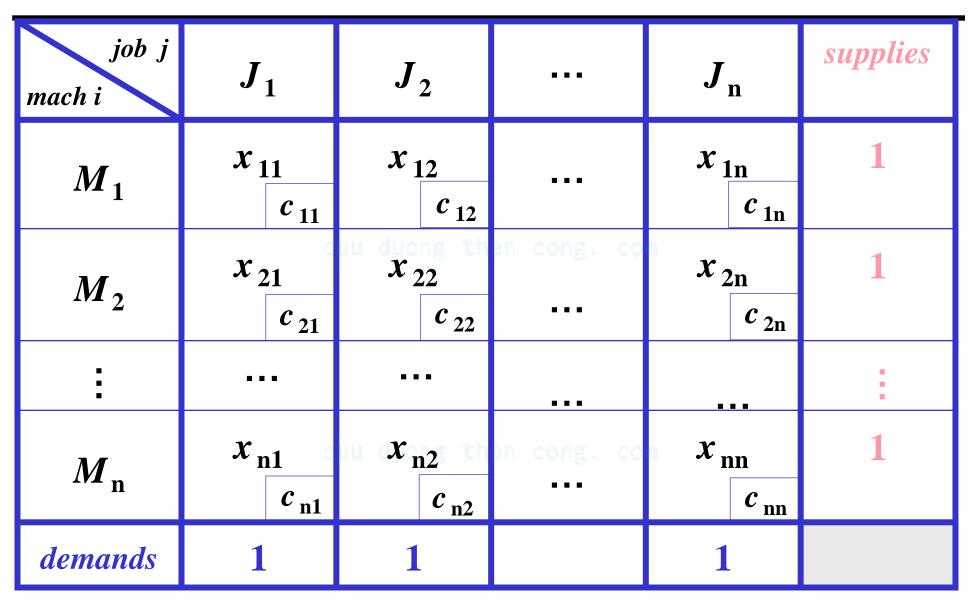
$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j$$

Thus, the assignment problem can be viewed as a special case of transportation problem

COST MATRIX



SIMPLIFIED COST MATRIX

Since demands and supplies are 1 for all assignment problems, we represent the assignment problem by the cost matrix below

job j mach i	${old J}_1^{{ m cuu}}$ d	ong t $m{J}_2^{_1}$ cons	. com •••	\boldsymbol{J}_n
\boldsymbol{M}_1	<i>c</i> ₁₁	<i>c</i> ₁₂	•••	<i>c</i> _{1<i>n</i>}
M_2	<i>c</i> ₂₁	<i>c</i> ₂₂	•••	C _{2n}
•••	•••	•••	•••	•••
M_n	<i>c</i> _{<i>n</i>1}	<i>c</i> _{<i>n</i>2}	•••	<i>c</i> _{<i>n</i>1}

HISTORY OF HUNGARIAN METHOD

□ First published by Harold Kuhn in 1955

cuu duong than cong. com

□ Based on earlier works of two Hungarian

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mathematicians, Dénes König and Jenő Egerváry

FACT

$$min Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$min \tilde{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k$$

$$s.t.$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad \forall i \text{ und}$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j$$

$$(i) \text{ und} \sum_{j=1}^{n} x_{ij} = 1 \quad \forall j$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j$$

$$(ii)$$

$$min \tilde{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k$$

$$(ii)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j$$

$$(ii)$$

$$min \tilde{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k$$

$$(ii)$$

$$\sum_{i=1}^{n} x_{ij} = 1 \quad \forall j$$

$$x_{ij} \in \{0, 1\} \quad \forall i, j$$

$$(ii)$$

$$min \tilde{Z} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k$$

$$x_{ij}^* \leq 1$$
 for $i, j \leq n$ also optimizes problem (ii)
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BASIC IDEA

The optimal assignment is not affected by a constant added or subtracted from any row of the original assignment cost matrix by the fact in the previous slide and

$$\tilde{Z} = \sum_{j=1}^{n} (c_{qj} - k) x_{qj} + \sum_{\substack{i=1 \ i \neq q}}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} - k \sum_{j=1}^{n} x_{qj}$$
$$= Z - k$$

A similar statement holds with respect to the column of the cost matrix

BASIC IDEA

□ If all elements of the cost matrix are nonnegative,

then the objective is nonnegative

cuu duong than cong. com

□ If the objective is nonnegative, and there exists a

feasible solution such that the total cost is zero,

then the feasible solution is the optimal solution

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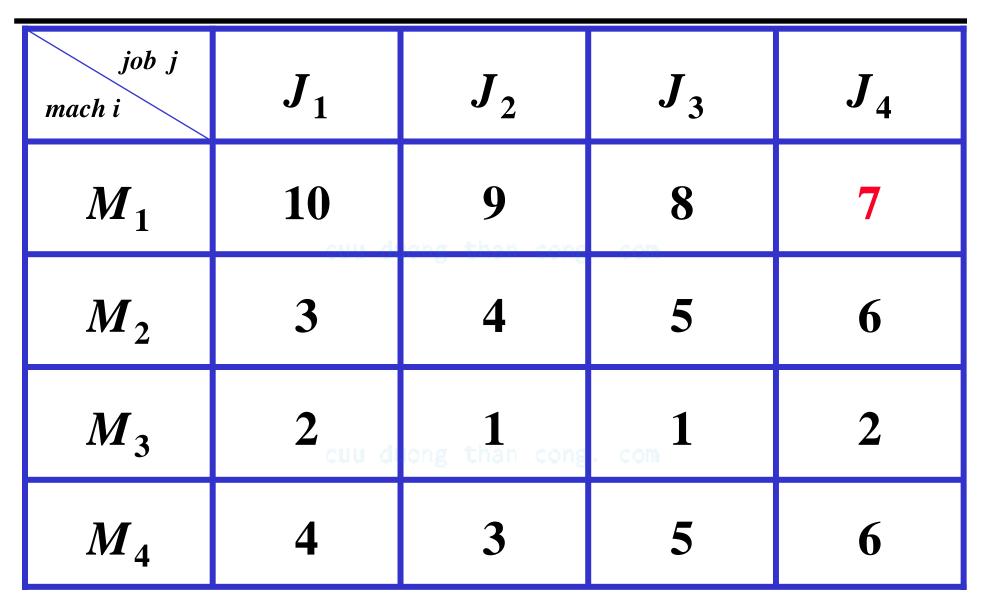
THE HUNGARIAN METHOD

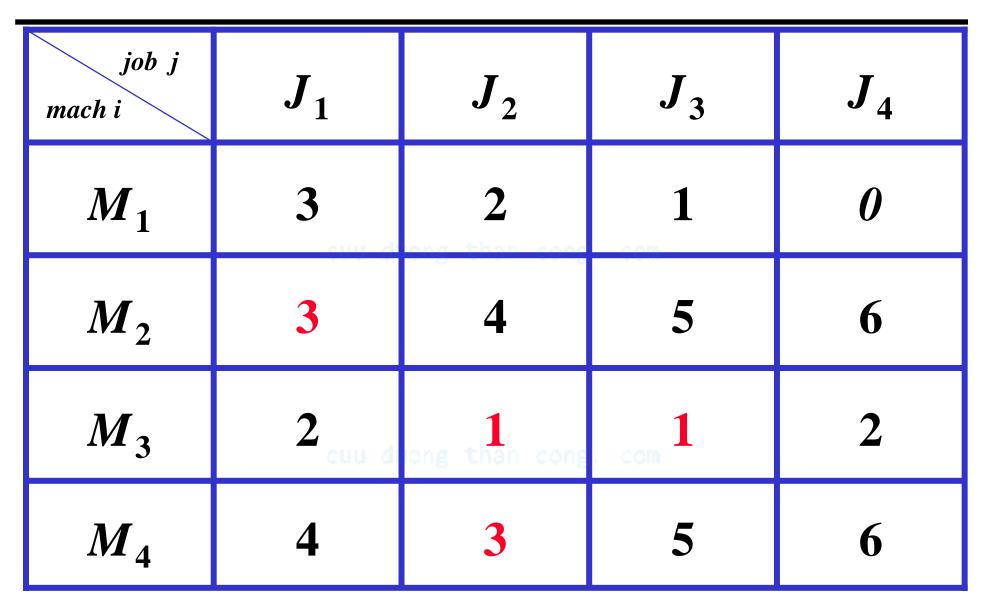
For each *i*, we consider the elements *i* and compute

$$\underline{c}_i = min\left(c_{ij}, 1 \le j \le n \right)$$

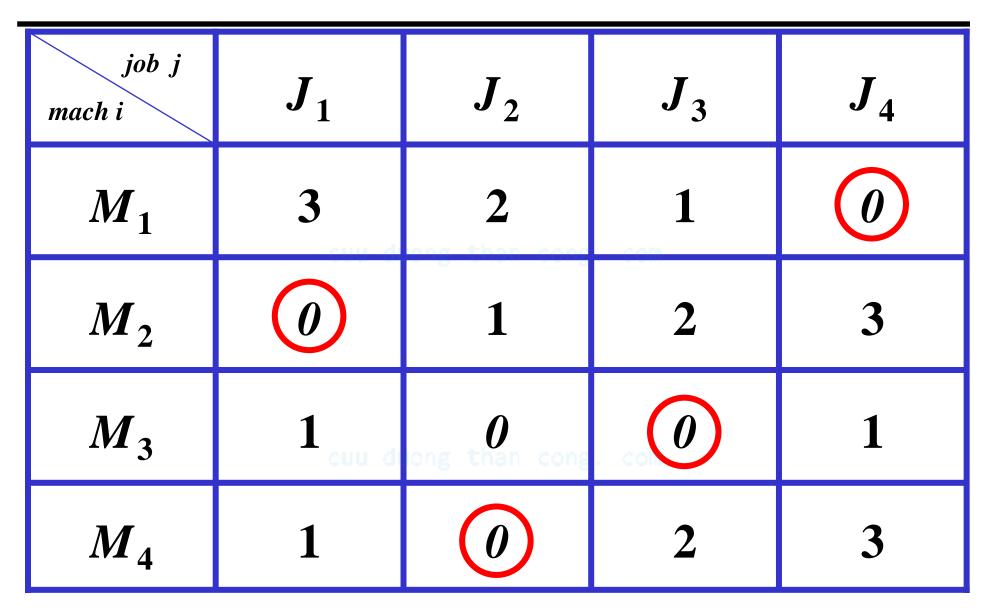
and subtract \underline{c}_i from each element in row *i* to get $\tilde{c}_{ij} = c_{ij} - \underline{c}_i, \ 1 \le j \le n$

- □ Then, we do the same procedure to each column
- We try to assign jobs only using the machines with zero costs since such an assignment, if found, is optimal





job j mach i	\boldsymbol{J}_1	\boldsymbol{J}_2	J ₃	${J}_4$
$\boldsymbol{M_1}$	3	2	1	0
M_2	0	1	2	3
<i>M</i> ₃	1 cuu di	O cong than cong	. com 0	1
M_4	1	0	2	3



HUNGARIAN METHOD

- □ In general, feasible assignment only using cells with zero costs may not exist after single row and column subtraction
- In such cases, we need to draw a minimum number of lines through certain rows and columns to cover all the cells with zero cost
- The minimum number of lines needed is the maximum number of jobs that can be assigned to
 - the zero cells subject to all the constraints, a
 - result that was proved by König

HUNGARIAN METHOD

□ Then we look up the submatrix that is not covered

by the lines to identify the smallest element

□ Subtract from each element of the submatrix the

value of the smallest element and add the value to

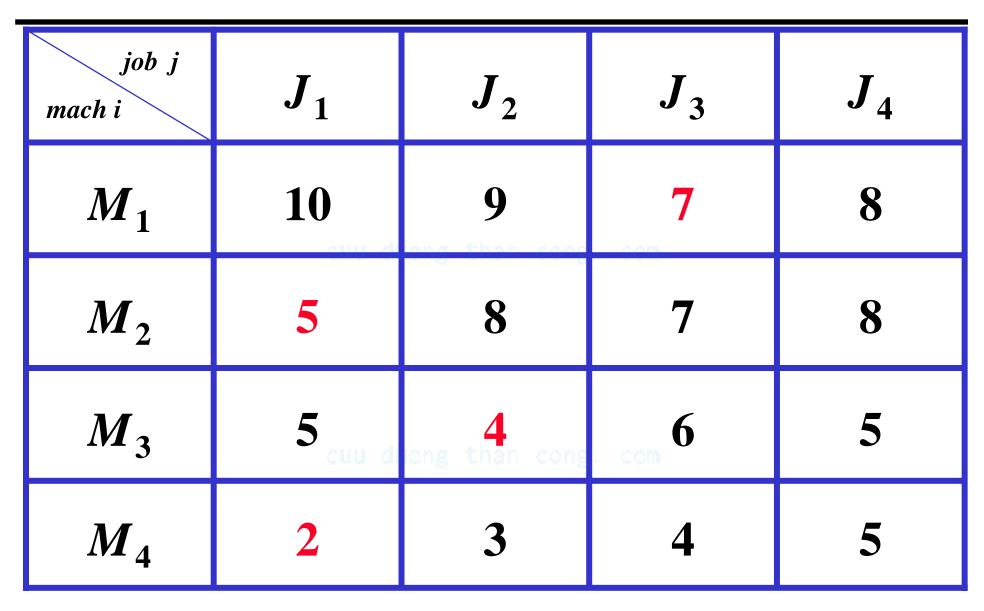
all elements at the intersection of two lines

HUNGARIAN METHOD

- □ The rationale for this operation is that we subtract
 - the smallest value from each element in a row
 - including any element that is covered by a line; to cur ducing than cong. com compensate we also need to add an equal value to
 - the element which is covered by the intersection
 - of two lines and therefore the operation keeps the value of the elements not at an intersection

unchanged

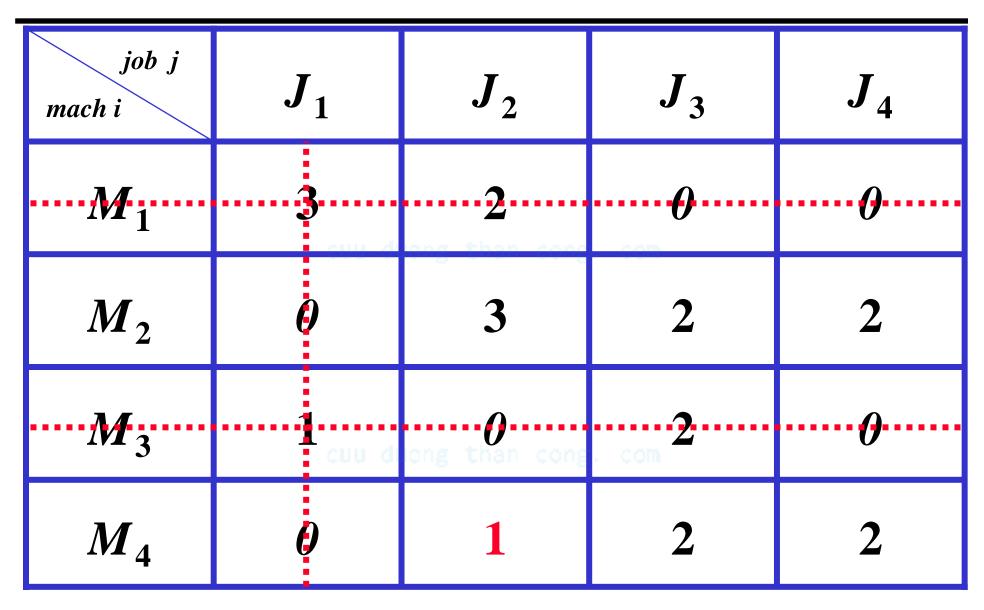
job j mach i	\boldsymbol{J}_1	\boldsymbol{J}_2	J ₃	${J}_4$
$\boldsymbol{M_1}$	10	9	7	8
M_2	5	8	7	8
<i>M</i> ₃	5 _{cuu d}	4 ong than cong	6 com	5
M_4	2	3	4	5

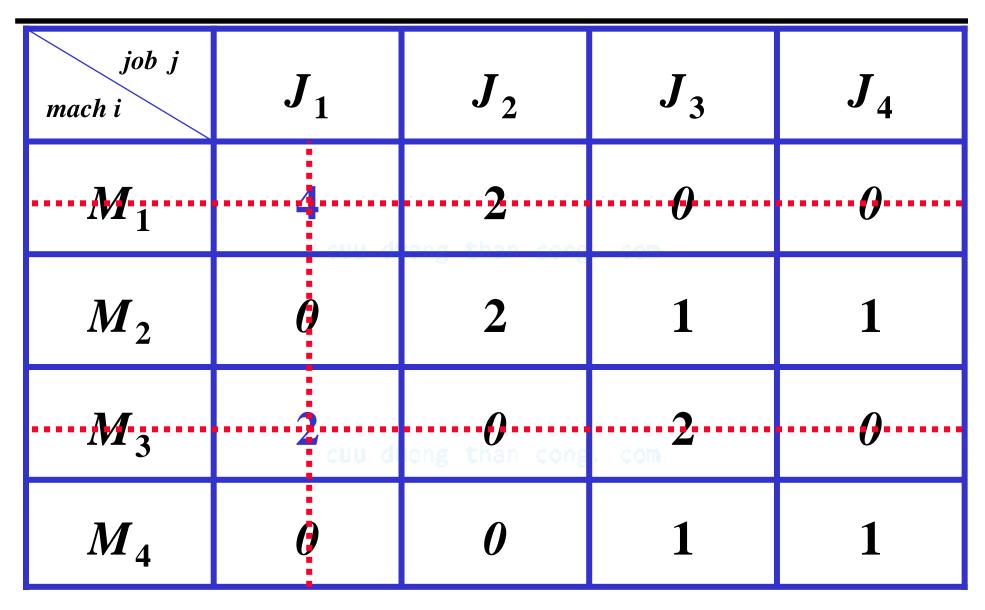


job j mach i	J_1	\boldsymbol{J}_2	J ₃	${J}_4$
$\boldsymbol{M_1}$	3	2	0	1
M_2	0	3	2	3
<i>M</i> ₃	1 cuu di	O ong than cong	2 com	1
M_4	0	1	2	3

job j mach i	\boldsymbol{J}_1	\boldsymbol{J}_2	J ₃	${old J}_4$
$\boldsymbol{M_1}$	3	2	0	1
M_2	0	3	2	3
<i>M</i> ₃	1 cuu di	ong than cong	2 com	1
M_4	0	1	2	3

job j mach i	J_1	\boldsymbol{J}_2	J ₃	J_4
$\boldsymbol{M_1}$	3	2	0	0
<i>M</i> ₂	0	3	2	2
<i>M</i> ₃	1 cuu di	O cong than cong	2 com	0
M_4	0	1	2	2





	J ₁	\boldsymbol{J}_2	J ₃	${old J}_4$
$\boldsymbol{M_1}$	4	2	0	0
M_2	0	2	1	1
<i>M</i> ₃	2 _{cuu d}	ong than cong	2 com	0
M_4	0	0	1	1

□ We cast the problem as an assignment with the

days being the machines and the courses being

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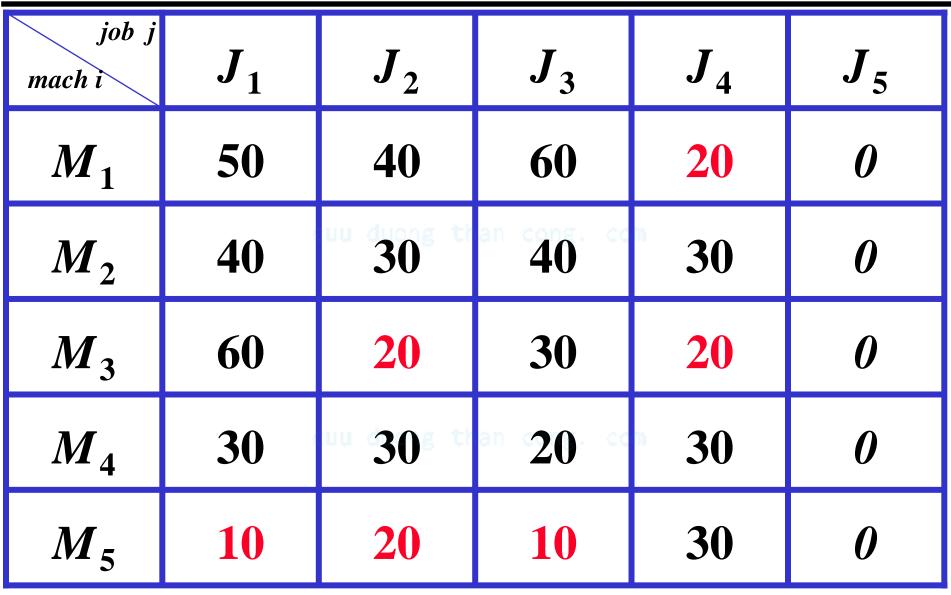
□ In order for the assignment problem to be

balanced, we introduce an additional course

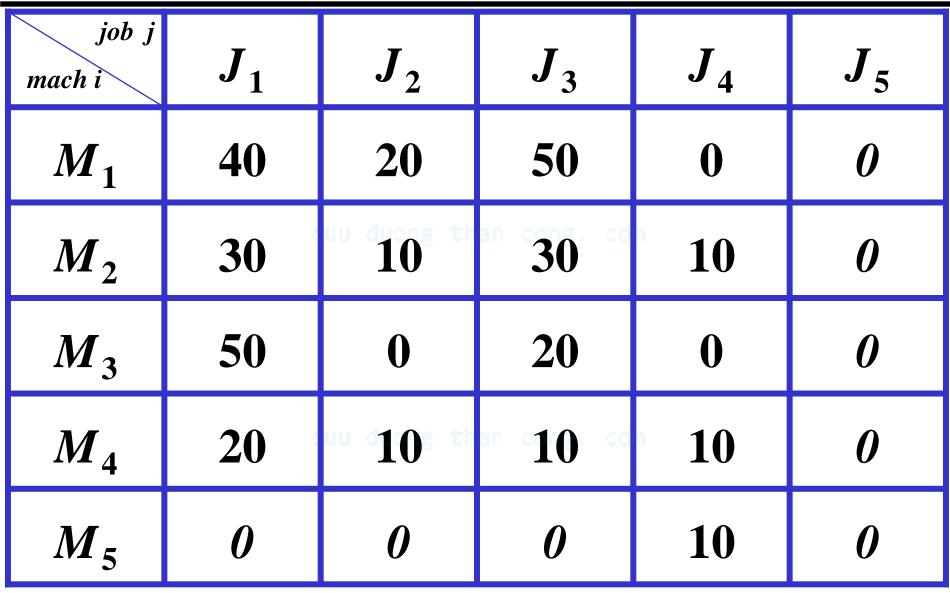
whose costs are zero for each day

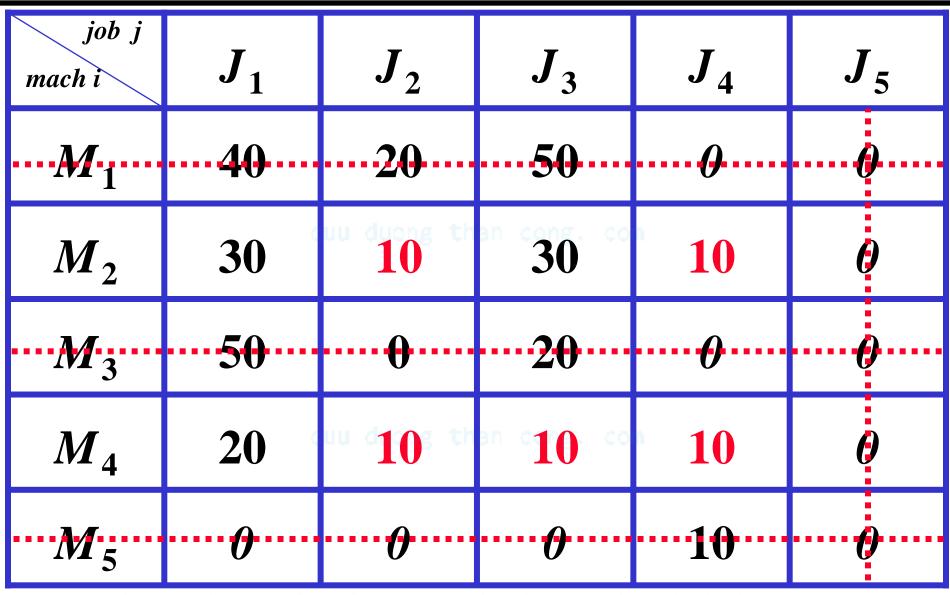
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the jobs



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