ECE 307 – Techniques for Engineering Decisions

Review of Combinatorial Analysis

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COMBINATORIAL ANALYSIS

- Many problems in probability theory may be solved by simply counting the number of ways a certain event may occur

- We review some basic aspects of combinatorial analysis

  - combinations
  - permutations

BASIC PRINCIPLE OF COUNTING

- Suppose that two experiments are to be performed:
  - experiment 1 may result in any one of the m possible outcomes
  - for each outcome of experiment 1, there exist n possible outcomes of experiment 2
- Therefore, there are mn possible outcomes of the two experiments

BASIC PRINCIPLE OF COUNTING

- **The basic principle is easy to prove the result by the use of exhaustive enumeration that the set of outcomes for the 2 experiments can be listed as:**

$$\begin{array}{ccccc} (1,1), & (1,2), & (1,3), & \cdots & (1,n) \\ (2,1), & (2,2), & (2,3), & \cdots & (2,n) \\ \vdots & & & & \\ (m,1), & (m,2), & (m,3), & \cdots & (m,n)\,, \end{array}$$

where, $(i\,,j)$ denotes outcome i in experiment 1 and outcome j in experiment 2

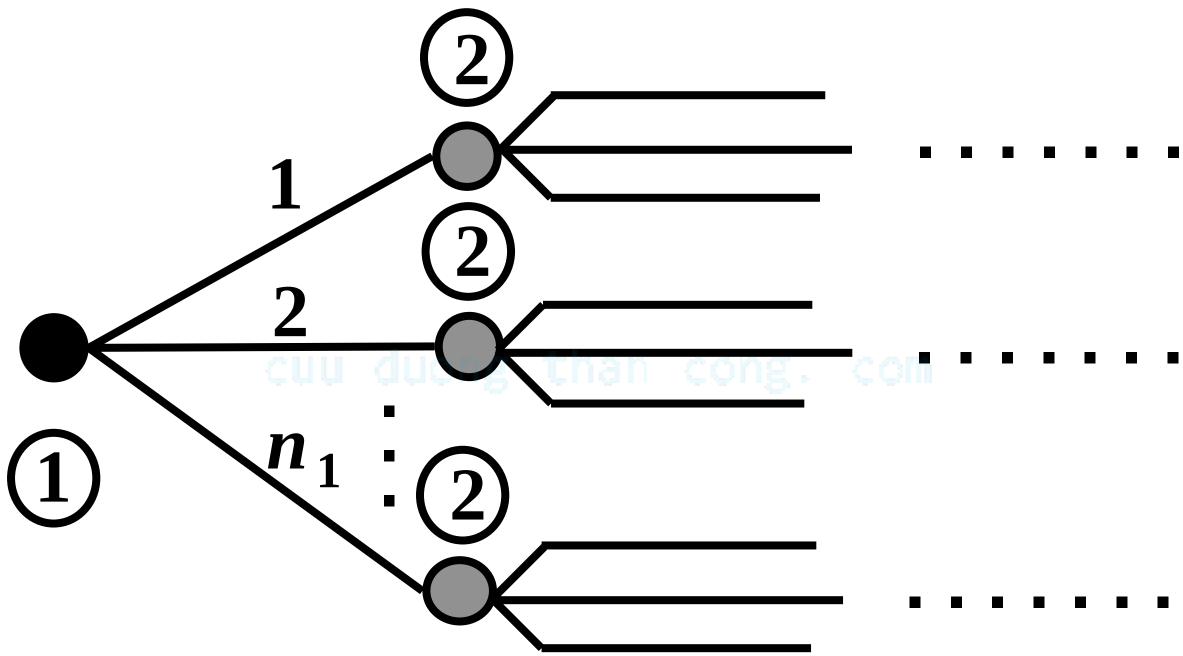
EXAMPLE 1: PAIR FORMATION

- Pairs need to be formed consisting of 1 boy and 1 girl by choosing from a group of 7 boys and 9 girls

- There exist $(7)(9) = 63$ possible pairs since there are 7 ways to pick a boy and 9 ways to pick a girl

GENERALIZED VERSION OF THE BASIC PRINCIPLE

- For r experiments with the first experiment having n_1 possible outcomes; for every outcome of the first experiment, there are n_2 possible outcomes for the second experiment, and so on



GENERALIZED VERSION OF THE BASIC PRINCIPLE

- There are

$$\prod_{i=1}^{r} n_i = n_1 \cdot n_2 \cdot n_3 \cdot \ldots \cdot n_r$$

possible outcomes for all the r experiments, i.e., there are $\prod_{i=1}^{r} n_i$ possible branches in the illustration – high dimensionality even for a moderate number of experiments

EXAMPLE 2: SUBCOMMITTEE CHOICES

- The executive committee of an *Engineering Open House* function consists of:
  - 3 first year students
  - 4 sophomores
  - 5 juniors
  - 2 seniors
- We need to form a subcommittee of 4 with each year represented:
- There are $3 \cdot 4 \cdot 5 \cdot 2 = 120$ different subcommittees

EXAMPLE 3: LICENSE PLATE

- We consider possible combinations for a six-place license plate with the first three places consisting of letters and the last three places of numbers

- Number of combinations with repeats allowed is

$$(26)\,(26)\,(26)\,(10)\,(10)\,(10) = 17{,}576{,}000$$

- Combination number if no repetition allowed is

$$(26)\,(25)\,(24)\,(10)\,(9)\,(8) = 11{,}232{,}000$$

EXAMPLE 4: *n* POINTS

- Consider n points at which we evaluate the

  function

$$f(i) \in \{0,1\}, \; i = 1, 2, \ldots, n$$

- Therefore, there are 2^n possible function values

PERMUTATIONS

- **A set of 3 objects$\{ A, B, C \}$ may be arranged in 6 different ways:**

BCA	ABC	CBA
BAC	ACB	CAB

- **Each arrangement is called a *permutation***
- **The total number of permutations is derived from the Basic Principle:**

  - **there are 3 ways to pick the first element**
  - **there are 2 ways to pick the second element**
  - **there is 1 way to pick the third element**

PERMUTATIONS

- **Therefore, there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange the 3 elements**

- **In general, a set of n objects can be arranged into**

$$n! = n\,(n-1)\,(n-2)\,\ldots\;1$$

different permutations

EXAMPLE 5: BASEBALL

- **Number of possible batting orders for a baseball team with nine members is**

$$9! = 362{,}880$$

- **Suppose that the team, however, has altogether 12 members; the number of possible batting orders is the product of the number of team formations and the number of permutations is**

$$\frac{12!}{3!\,9!} \bullet 9! = \frac{12!}{3!} = 2(11!) = 79{,}833{,}600$$

EXAMPLE 6: CLASSROOM

- A class with 6 boys and 4 girls is ranked in terms of weight; assume that no two students have the same weight
- There are

$$10! = 3,628,800$$

possible rankings

- If the boys (girls) are ranked among themselves, the number of different possible rankings is

$$6!4! = 17,280$$

EXAMPLE 7: BOOKS

- **A student has 10 books to put on the shelf:**

  4 *EE*, 3 *Math*, 2 *Econ*, and 1 *Decision*

- **Student arranges books so that all books in each category are grouped together**

- **There are 4!3!2!1! arrangements so that all *EE* books are first in line, then the *Math* books, *Econ* books, and *Decision* book**

EXAMPLE 8: BOOKS

- But, there are 4! possible orderings of the subjects

- Therefore, there are

$$4!4!3!2!1! = 6912$$

permutations of arranging the 10 books

EXAMPLE 9: PEPPER

- We wish to determine the number of different letter arrangements in the word $PEPPER$
- Consider first the letters $P_1 E_1 P_2 P_3 E_2 R$ where we distinguish the repeated letters among themselves: there are $6!$ permutations of the 6 distinct letters

EXAMPLE 9: PEPPER

- However, if we consider any single permutation of the 6 letters – for example, $P_1 P_2 E_1 P_3 E_2 R$ – provides the same word $PPEPER$ as 11 other permutations if we fail to distinguish between the same letters
- Therefore, there are $6!$ permutations for distinct letters but only

$$\frac{6!}{3!2!} = 60$$

permutations when repeated letters are not distinct

GENERAL STATEMENT

- Consider a set of n objects in which

n_1	are alike (category 1)
n_2	are alike (category 2)
$\vdots$	
n_r	are alike (category r)

- There are

$$\frac{n!}{n_1!n_2!.....n_r!}$$

different permutations

EXAMPLE 9: COLORED BALLS

- **We have 3 *white*, 4 *red*, and 4 *black* balls which we arrange in a row; similarly colored balls are indistinguishable from each other**
- **There are**

$$\frac{11!}{3!4!4!} = 11{,}550$$

possible less arrangements

COMBINATIONS

- Given n objects, we form groups of r objects and determine the number of different groups we can form

- For example, consider 5 objects denoted as A, B, C, D and E and form groups of 3 objects:
  - we can pick the first item in exactly 5 ways
  - we can pick the second item in exactly 4 ways
  - we can pick the third item in exactly 3 ways

COMBINATIONS

and, therefore, we can select

$$5 \cdot 4 \cdot 3 = 60$$

possible groups in which the order of the groups is taken into account

- But, if the order of the objects is not of interest, i.e., we ignore that each group of three objects can be arranged in 6 different permutations, the total number of distinct groups is

$$\frac{5!}{2!3!} = \frac{60}{6} = 10$$

GENERAL STATEMENT ON COMBINATIONS

- **The objective is to arrange n elements into groups of r elements**
- **We can select groups of r elements**

$$\frac{n!}{(n-r)!}$$

different ways

- **But, each group of r has $r!$ permutations**
- **The number of different combinations is**

$$\frac{n!}{(n-r)!r!}$$

BINOMIAL COEFFICIENTS

- ❑ We define the term

$$\binom{n}{r} \triangleq \frac{n!}{(n-r)!\,r!}$$

as the *binomial coefficient* of n and r

- ❑ A binomial coefficient gives the number of possible combinations of n elements taken r at a time

EXAMPLE 10: COMMITTEE SELECTION

- We wish to select three persons to represent a class of 40: how many groups of 3 can be formed?

- There are

$$\frac{40!}{37!3!} = \frac{40 \cdot 39 \cdot 38}{3 \cdot 2 \cdot 1} = 20 \cdot 13 \cdot 38 = 9880$$

possible committee selections

EXAMPLE 11: GROUP FORMATION

- Given a group of 5 *boys* and 7 *girls*, form sets consisting of 2 *boys* and 3 *girls*

- There are

$$\binom{5}{2}\binom{7}{3} = \frac{5!}{3!2!}\frac{7!}{4!3!} = \frac{5 \cdot 4}{2}\ \frac{7 \cdot 6 \cdot 5}{3 \cdot 2} = 350$$

possible ways to form such groups

GENERAL COMBINATORIAL IDENTITY

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

number of ways of selecting groups of r from n

number of ways of selecting groups of $r-1$ from $n-1$

number of ways of selecting groups of r from $n-1$

MULTINOMIAL COEFFICIENTS

- Given a set of n distinct items, form r distinct groups of respective sizes n_1, n_2, . . . , and n_r with

$$\sum_{i=1}^{r} n_i = n$$

- There are

$$\binom{n}{n_1}$$

possible choices for the first group

MULTINOMIAL COEFFICIENTS

- For each choice of the first group, there are

$$\binom{n-n_1}{n_2}$$

possible choices for the second group

- We continue with this reasoning and we conclude that there are

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

possible groups

MULTINOMIAL COEFFICIENTS

- The previous conclusion was gained by realizing that

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots n_{r-1}}{n_r}=$$

$$\frac{n!}{(n-n_1)!n_{1!}}\,\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\cdots\frac{n-n_1-n_2-\ldots n_{r-1}}{0!n_r!}=$$

$$\frac{n!}{n_1!n_2!\ldots n_r!}$$

MULTINOMIAL COEFFICIENTS

- Let

$$n = n_1 + n_2 + n_3 + \ldots + n_r$$

we define the *multinomial coefficient*

$$\binom{n}{n_1, n_2, \ldots, n_r} \triangleq \frac{n!}{n_1! n_2! n_3! \ldots n_r!}$$

- A multinomial coefficient represents the number of possible divisions of n distinct objects into r distinct groups of respective sizes $n_1, n_2, \ldots, n_r$

EXAMPLE 12: POLICE

- A police department of a small town has 10 officers
- The department policy is to have 5 members on street patrol, 2 members at the station and 3 on reserve

- The number of possible divisions is

$$\frac{10!}{5!3!2!} = 2{,}520$$

EXAMPLE 13: TEAM FORMATION

- We need to form two teams, the *A* team and the *B* team, with each team having 5 *boys* from a group of 10 *boys*
- There are

$$\frac{10!}{5!5!} = 252$$

possible divisions

EXAMPLE 13: TEAM FORMATION

- Suppose that these two teams are to play against one another
- In this case, the order of the two teams is irrelevant since there is no team A and team B per se but simply a division of 10 *boys* into 2 groups of 5 each
- The number of ways to form the two teams is

$$\frac{1}{2!}\left(\frac{10!}{5!5!}\right) = 126$$

EXAMPLE 14: TEA PARTY

- A woman has 8 friends of whom she will invite 5 to a tea party
- How many choices does she have if 2 of the friends are feuding and refuse to attend together?
- How many choices does she have if 2 of her friends will only attend together?