

Chương 7: DFT & FFT

•DTFT

•DFT

•FFT

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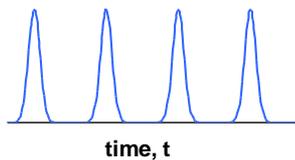
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ThS. Nguyễn Thanh Tuấn

Khoa Điện-Điện tử, Đại học Bách Khoa TP.HCM

nttbk97@yahoo.com

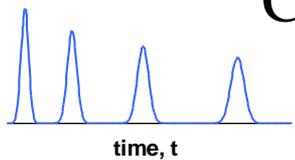
Miền thời gian

Miền tần số



time, t

Continuous



time, t

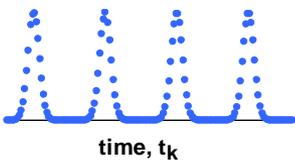
Periodic **FS**
 (period T)
 Aperiodic **FT**

Discrete

Continuous

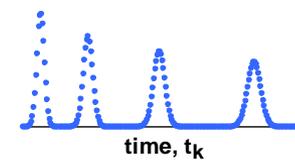
$$c_k = \frac{1}{T} \cdot \int_0^T s(t) \cdot e^{-jk\omega t} dt$$

$$S(f) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j2\pi f t} dt$$



time, t_k

Discrete



time, t_k

Periodic **DFS**
 (period T)
 Aperiodic **DTFT**
DFT

Discrete

Continuous

Discrete

$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi kn}{N}}$$

$$S(f) = \sum_{n=-\infty}^{+\infty} s[n] \cdot e^{-j2\pi f n}$$

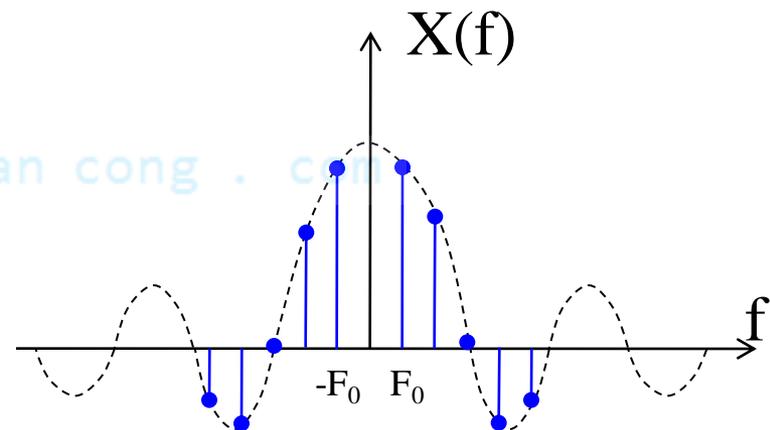
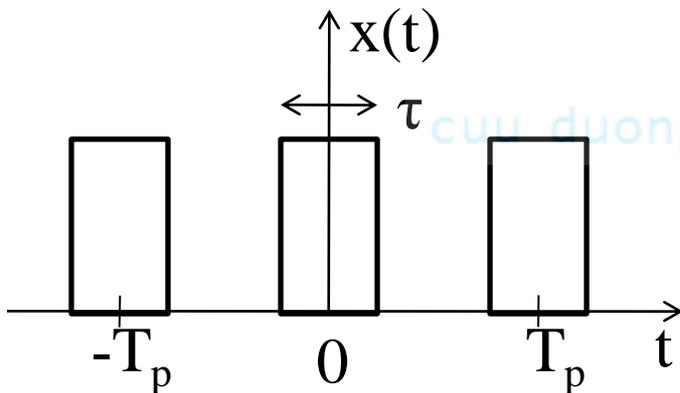
$$\tilde{c}_k = \frac{1}{N} \sum_{n=0}^{N-1} s[n] \cdot e^{-j\frac{2\pi kn}{N}}$$

Chuỗi Fourier (Fourier series-FS)

- Tín hiệu $x(t)$ tuần hoàn, chu kỳ T_p , tần số $F_0 = 1/T_p$

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k e^{j2\pi k F_0 t}$$

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

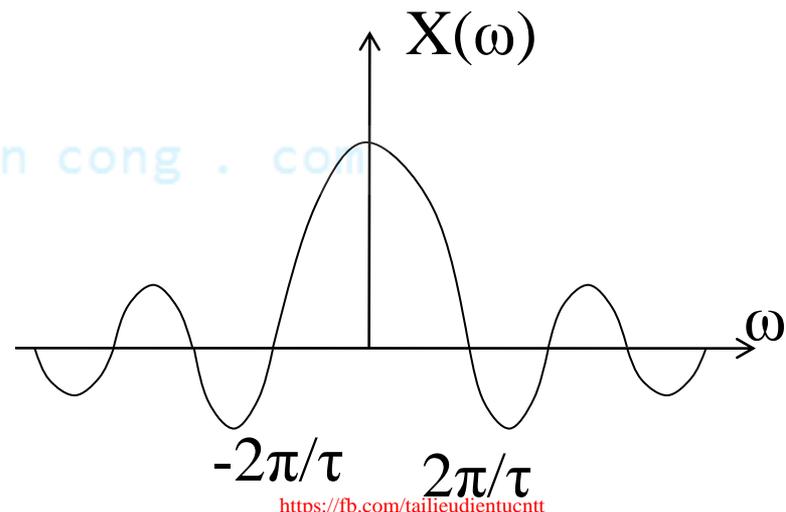
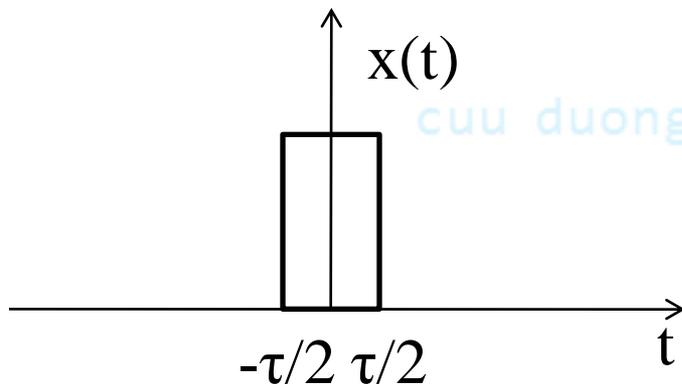


Biến đổi Fourier (Fourier transform-FT)

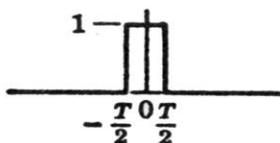
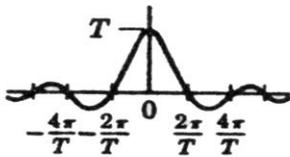
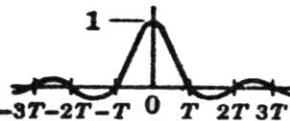
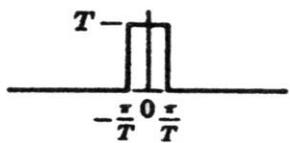
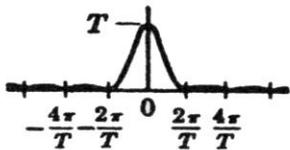
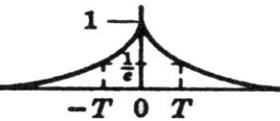
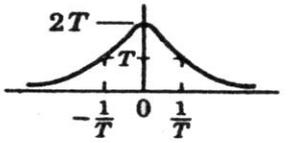
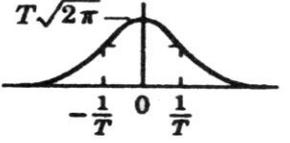
- Tín hiệu $x(t)$ **không tuần hoàn**

$$x(t) = \int_{-\infty}^{+\infty} X(f) e^{j2\pi ft} df$$

$$X(f) = \int_{-\infty}^{+\infty} x(t) e^{-j2\pi ft} dt$$



Biến đổi Fourier của một số tín hiệu cơ bản

	$f(t) = \int_{-\infty}^{\infty} F_F(i\omega) e^{i\omega t} \frac{d\omega}{2\pi}$	$F_F(i\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$	
	$\text{rect} \frac{t}{T} = \begin{cases} 1 & (t < T/2) \\ 0 & (t > T/2) \end{cases}$	$T \text{sinc} \frac{\omega T}{2\pi} \equiv T \frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}}$	
	$\text{sinc} \frac{t}{T} \equiv \frac{\sin \frac{\pi t}{T}}{\frac{\pi t}{T}}$	$T \text{rect} \frac{\omega T}{2\pi} = \begin{cases} 0 & (\omega < \frac{\pi}{T}) \\ T & (\omega > \frac{\pi}{T}) \end{cases}$	
	$\begin{cases} 1 - \frac{ t }{T} & (t < T) \\ 0 & (t \geq T) \end{cases}$	$T \text{sinc}^2 \frac{\omega T}{2\pi} \equiv T \left(\frac{\sin \frac{\omega T}{2}}{\frac{\omega T}{2}} \right)^2$	
	$e^{-\frac{ t }{T}}$	$\frac{2T}{(\omega T)^2 + 1}$	
	$e^{-\frac{1}{2} \left(\frac{t}{T}\right)^2}$	$\sqrt{2\pi} T e^{-\frac{1}{2} (\omega T)^2}$	

7.1 DTFT

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

- Định nghĩa
- Hạn chế tính toán
 - Chiều dài $x(n)$ vô hạn
 - $X(\omega)$ liên tục
- Cửa sổ hóa tín hiệu
 - Hàm cửa sổ
 - Ảnh hưởng
- Tính toán DTFT
 - Tại 1 tần số
 - Trong khoảng tần số \rightarrow không thể

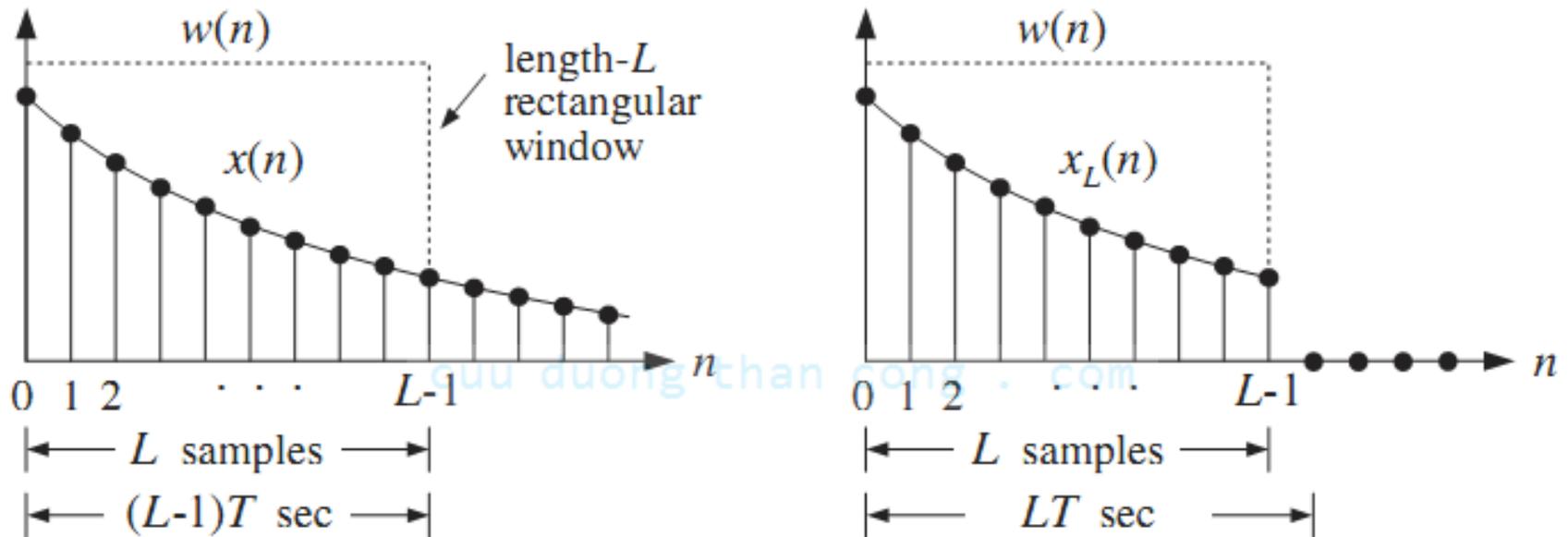
$$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n) e^{j\omega n}$$

$$\hat{X}(f) = \sum_{n=-\infty}^{\infty} x(nT) e^{-2\pi jfnT}$$

$$\hat{X}_L(f) = \sum_{n=0}^{L-1} x(nT) e^{-2\pi jfnT}$$

$$T\hat{X}(f) = X(f) + X(f - f_s) + X(f + f_s) + \dots, \quad -\frac{f_s}{2} \leq f \leq \frac{f_s}{2}$$

Cửa sổ hóa thời gian



$$x_L(n) = x(n)w(n) \quad X_L(\omega) = \int_{-\pi}^{\pi} X(\omega')W(\omega - \omega') \frac{d\omega'}{2\pi}$$

- Giảm độ phân giải tần số \leftarrow chiều dài cửa sổ
- Gây rò rỉ tần số \leftarrow hình dạng cửa sổ

$$\Delta f = 1/T_L$$

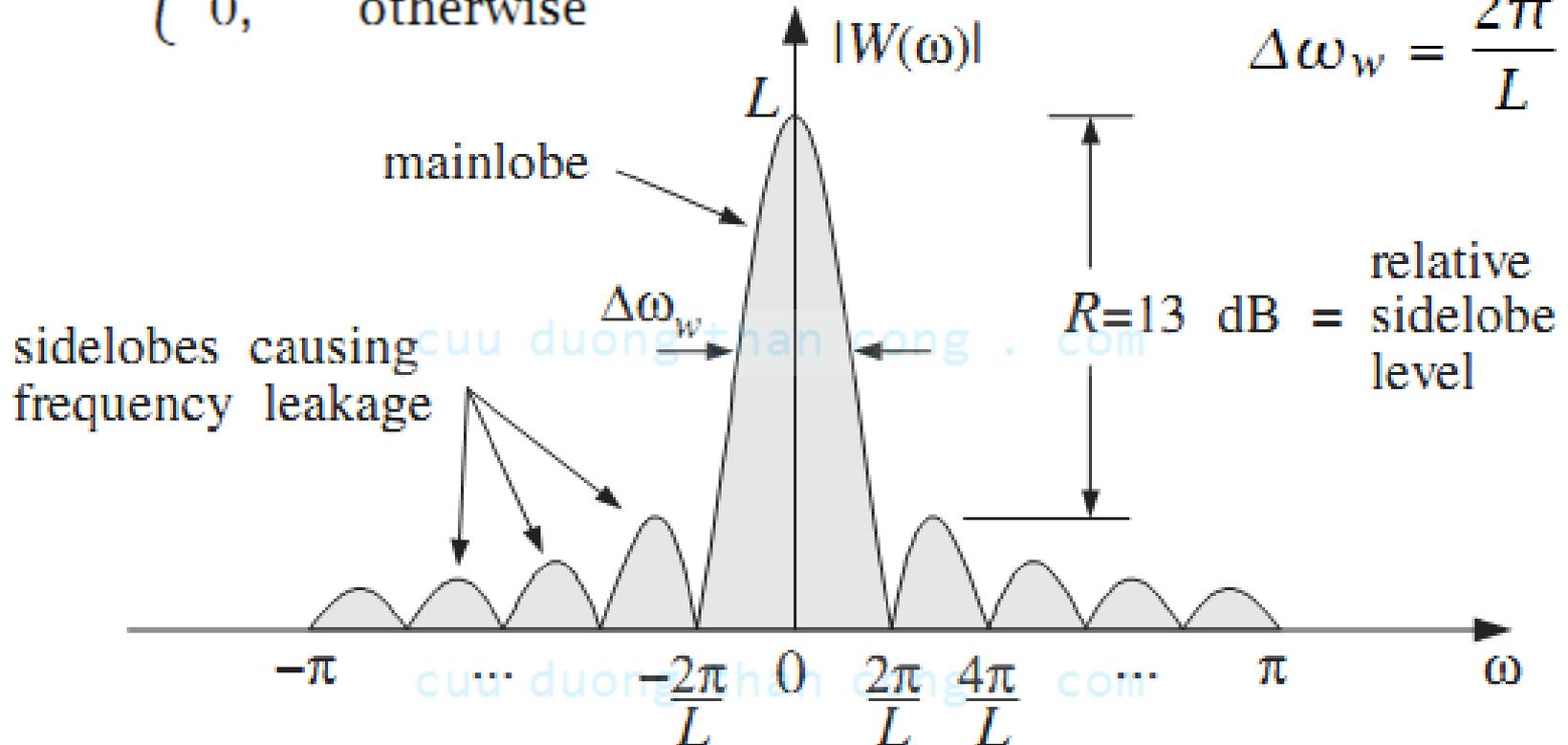
$$W(\omega) = \sum_{n=0}^{L-1} w(n)e^{-j\omega n}$$

$$T_L = LT$$

Cửa sổ chữ nhật

$$w(n) = \begin{cases} 1, & \text{if } 0 \leq n \leq L - 1 \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta\omega_w = \frac{2\pi}{L}$$



$$W(\omega) = \frac{1 - e^{-jL\omega}}{1 - e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2}$$

Tín hiệu tuần hoàn lấy mẫu

$$x(t) = e^{2\pi j f_1 t}, \quad -\infty < t < \infty$$

$$x(n) = e^{2\pi j f_1 n T} = e^{j\omega_1 n}, \quad -\infty < n < \infty$$

$$\omega_1 = 2\pi T f_1 = 2\pi f_1 / f_s$$

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$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-2\pi j f t} dt = \int_{-\infty}^{\infty} e^{-2\pi j (f - f_1) t} dt = \delta(f - f_1)$$

$$X(\omega) = \hat{X}(f) = \frac{1}{T} X(f) = \frac{1}{T} \delta(f - f_1) \quad -f_s/2 \leq f \leq f_s/2$$

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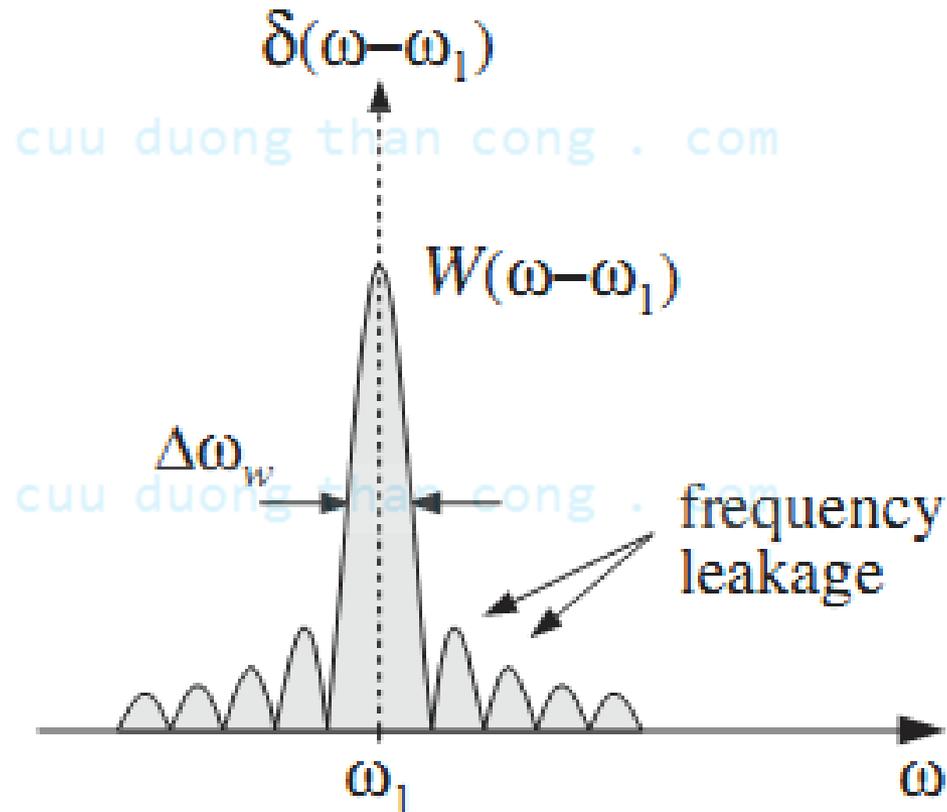
$$X(\omega) = 2\pi \delta(\omega - \omega_1) \quad -\pi \leq \omega \leq \pi \quad |a| \delta(ax) = \delta(x)$$

$$x(n) = \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} \frac{d\omega}{2\pi} = \int_{-\pi}^{\pi} 2\pi \delta(\omega - \omega_1) e^{j\omega n} \frac{d\omega}{2\pi} = e^{j\omega_1 n}$$

Ảnh hưởng của cửa sổ chữ nhật

$$x_L(n) = e^{j\omega_1 n}, \quad n = 0, 1, \dots, L - 1$$

$$X_L(\omega) = W(\omega - \omega_1)$$



Ảnh hưởng của cửa sổ chữ nhật (2)

$$x(t) = A_1 e^{2\pi j f_1 t} + A_2 e^{2\pi j f_2 t}, \quad -\infty < t < \infty$$

$$X(f) = A_1 \delta(f - f_1) + A_2 \delta(f - f_2)$$

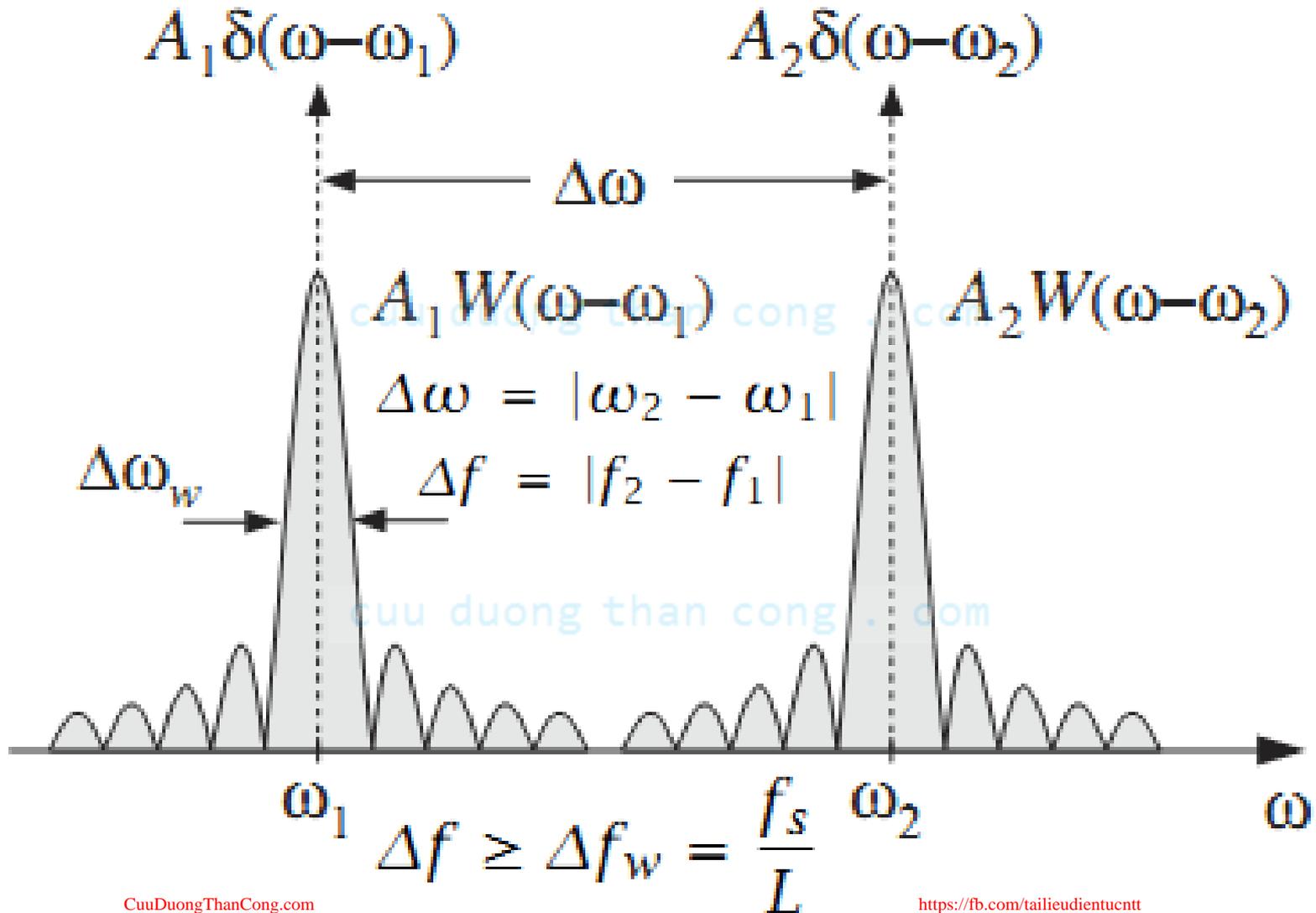
$$x(n) = A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n}, \quad -\infty < n < \infty$$

$$X(\omega) = 2\pi A_1 \delta(\omega - \omega_1) + 2\pi A_2 \delta(\omega - \omega_2), \quad -\pi \leq \omega \leq \pi$$

$$x_L(n) = A_1 e^{j\omega_1 n} + A_2 e^{j\omega_2 n}, \quad 0 \leq n \leq L - 1$$

$$X_L(\omega) = A_1 W(\omega - \omega_1) + A_2 W(\omega - \omega_2)$$

Ảnh hưởng của cửa sổ chữ nhật (2)



Các cửa sổ khác

- Tăng độ phân giải tần số: búp chính rộng hơn (và thấp hơn) $c \geq 1$

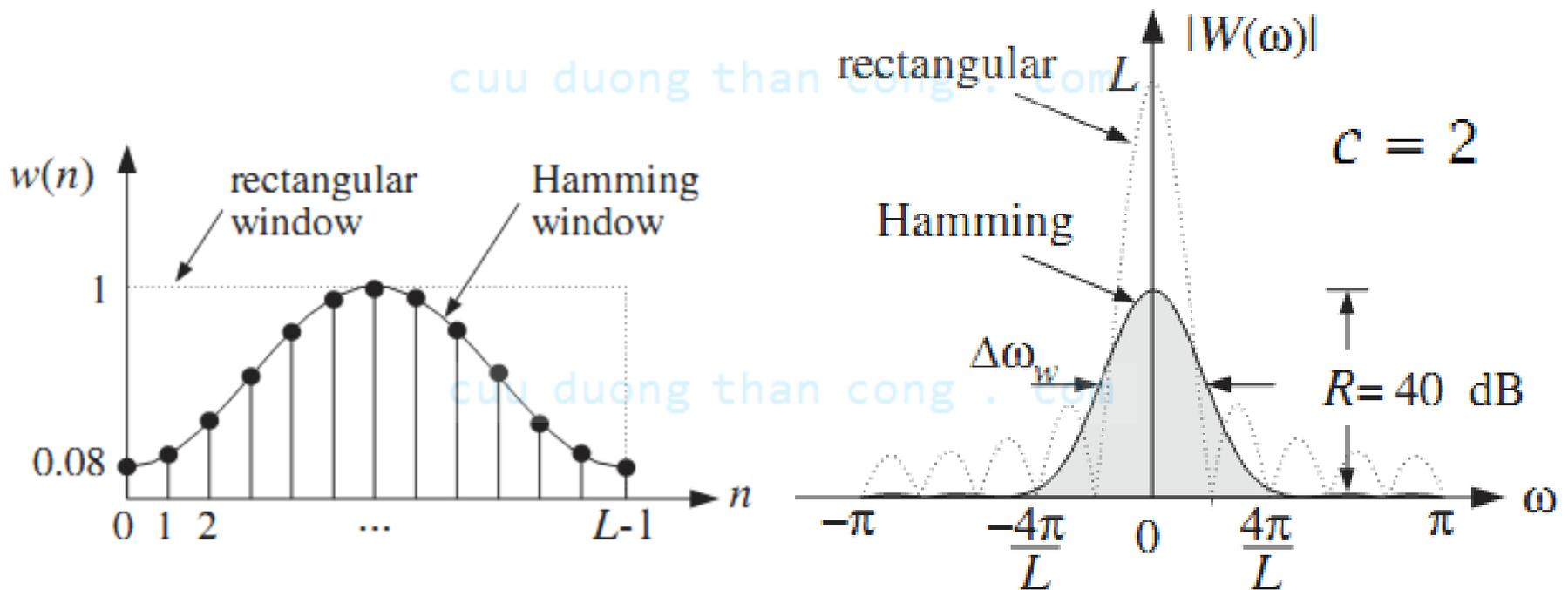
$$\Delta f_w = c \frac{f_s}{L} = c \frac{1}{T_L \text{com}}$$

$$\Delta \omega_w = c \frac{2\pi}{L}$$

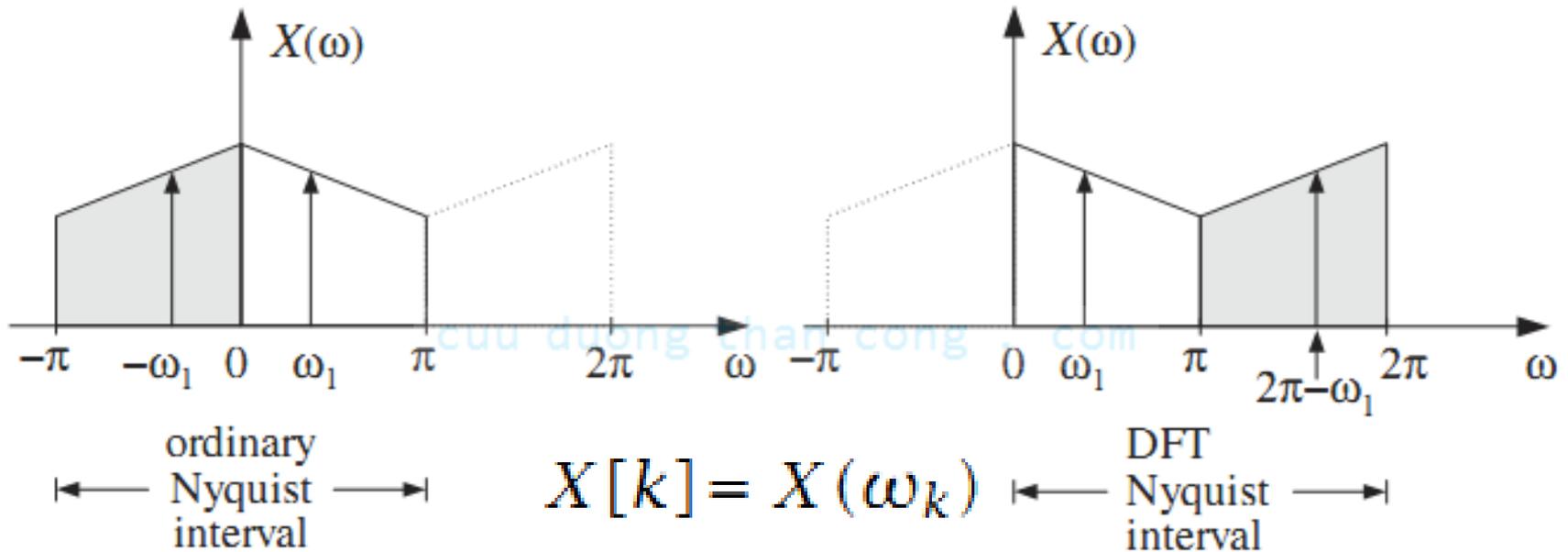
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Cửa sổ Hamming

$$w(n) = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{L-1}\right), & \text{if } 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases}$$



Tính toán DTFT



$$\cos(\omega_1 n) \quad \{ \omega_1, -\omega_1 \} \Leftrightarrow \{ \omega_1, 2\pi - \omega_1 \}$$

$$\omega_k = \omega_a + k \frac{\omega_b - \omega_a}{N} = \omega_a + k \Delta\omega_{\text{bin}}, \quad k = 0, 1, \dots, N - 1$$

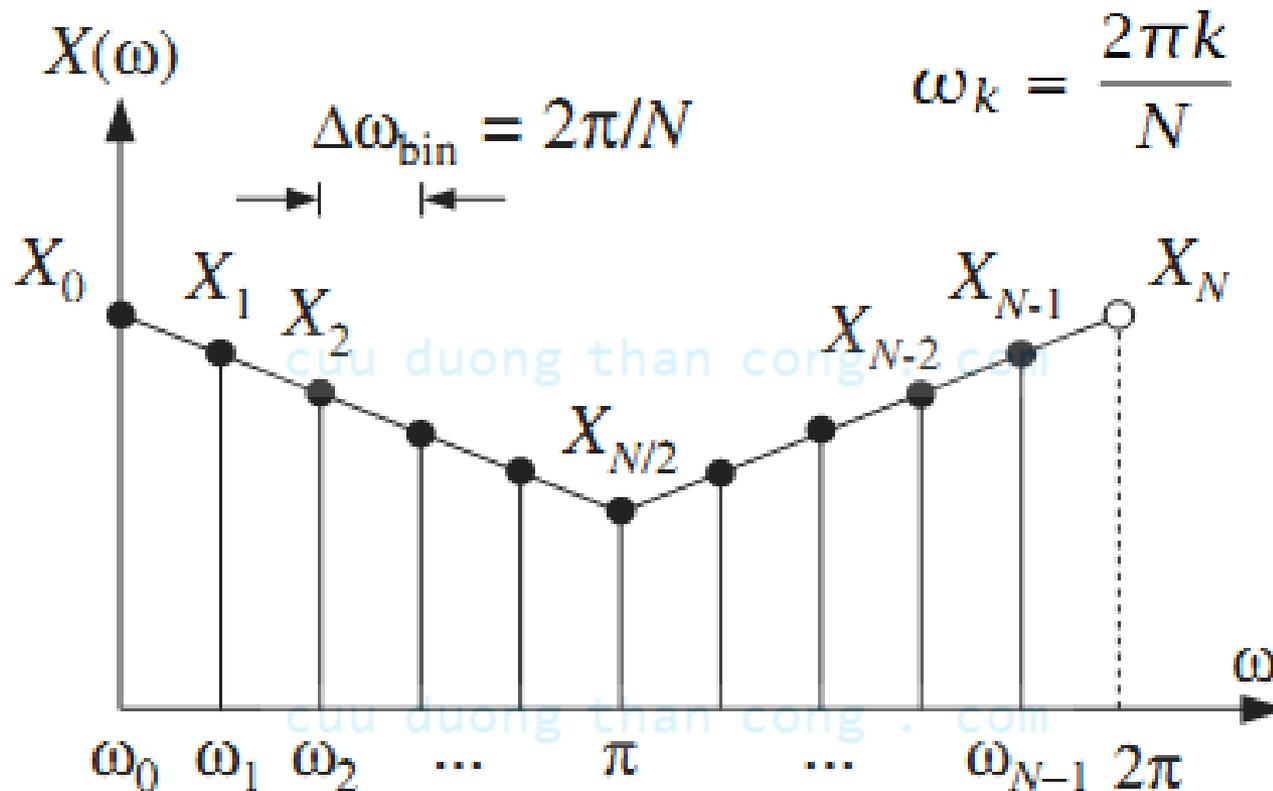
7.2 DFT

- Định nghĩa: DFT N điểm
- Quan hệ của DFT với biến đổi Z
- Dạng ma trận
- Chèn zero
- Giảm modulo-N
- DFT ngược (IDFT)
- Điều kiện khôi phục đúng

$$X(k) = \sum_{n=0}^{L-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, 2, \dots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N}, \quad n = 0, 1, 2, \dots, N-1$$

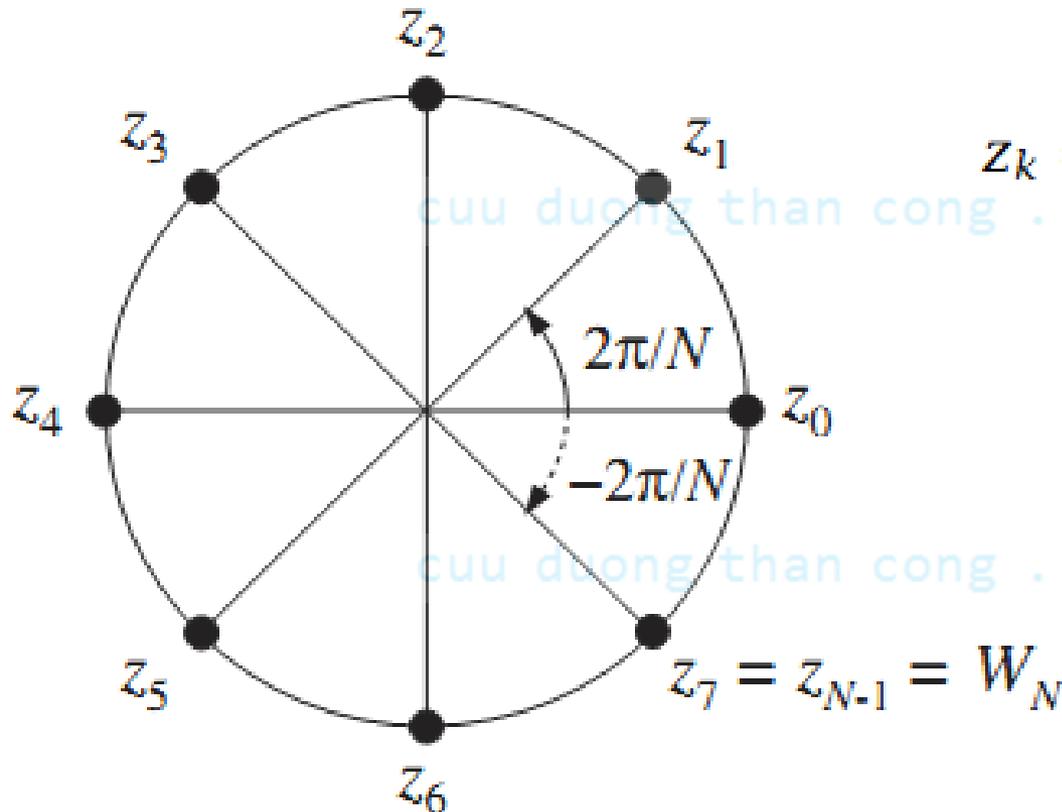
DFT N điểm



$$X(k + N) = X(\omega_{k+N}) = X(\omega_k + 2\pi) = X(\omega_k) = X(k)$$

DFT và biến đổi Z

$$X(\omega_k) = X(z_k) = \sum_{n=0}^{L-1} x(n) z_k^{-n}$$



$$z_k = e^{j\omega_k} = e^{2\pi jk/N}$$

$$z^N = 1$$

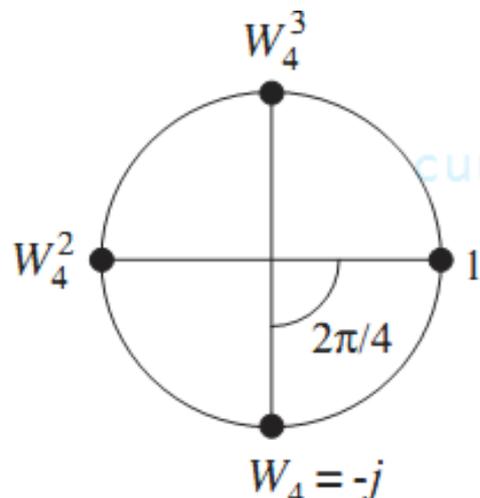
$$z_7 = z_{N-1} = W_N$$

DFT dạng ma trận

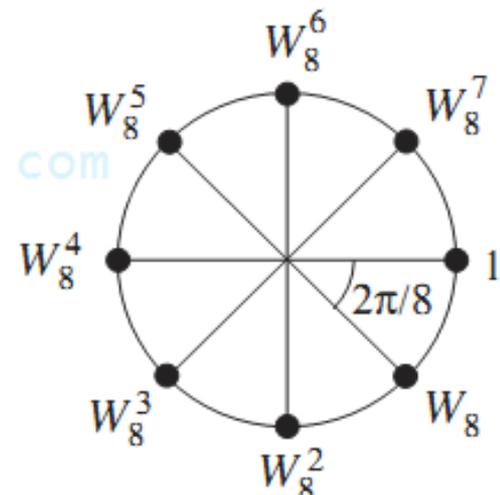
$$\mathbf{X} = \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{N-1} \end{bmatrix} \quad \mathbf{X} = \text{DFT}(\mathbf{x}) = \mathbf{A}\mathbf{x} \quad \mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{L-1} \end{bmatrix}$$

$$X_k = \sum_{n=0}^{L-1} A_{kn} x_n$$

$$A_{kn} = e^{-j\omega_k n} = e^{-2\pi j k n / N} = W_N^{kn}$$



$$W_N = e^{-2\pi j / N}$$



Ma trận DFT

$$A = \begin{bmatrix} 1 & 1 \\ 1 & W_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} X_0 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} x_0 + x_1 \\ x_0 - x_1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Chèn zero

- Khi $L < N$

$$\mathbf{x}_D = [x_0, x_1, \dots, x_{L-1}, \underbrace{0, 0, \dots, 0}_{D \text{ zeros}}] \quad X_D(\omega) = X(\omega)$$

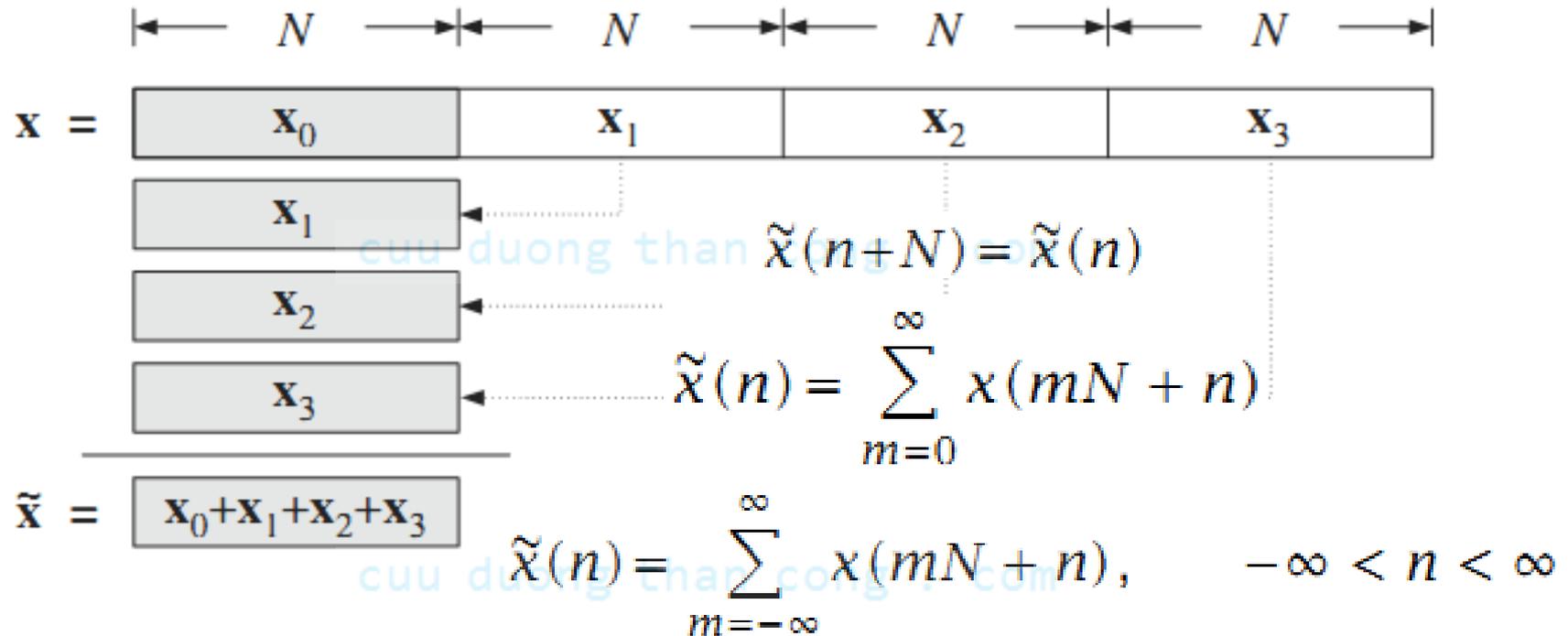
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$$\mathbf{x}_D = [\underbrace{0, 0, \dots, 0}_{D \text{ zeros}}, x_0, x_1, \dots, x_{L-1}] \quad X_D(\omega) = e^{-j\omega D} X(\omega)$$

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Giảm modulo-N

- Khi $L > N$



$$\tilde{X} = X = AX = \tilde{A}\tilde{X}$$

$$\tilde{A}_{kn} = W_N^{kn}$$

$$0 \leq k \leq N - 1, \quad 0 \leq n \leq N - 1$$

$$A = [\tilde{A}, \tilde{A}, \tilde{A}, \dots]$$

Ví dụ

$$\mathbf{X} = \mathbf{A}\mathbf{x} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j & 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \\ 3 \\ 4 \\ -2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 + 4j \\ -2 \\ 8 - 4j \end{bmatrix}$$

$$\tilde{\mathbf{X}} = \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ -1 & 1 & -1 & -j \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 + 4j \\ -2 \\ 8 - 4j \end{bmatrix}$$

Tính không duy nhất của DFT

$$\begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \\ X_6 \\ X_7 \end{bmatrix}, \begin{bmatrix} X_0 + X_4 \\ X_1 \\ X_2 \\ X_3 \\ 0 \\ X_5 \\ X_6 \\ X_7 \end{bmatrix}, \begin{bmatrix} X_0 + X_4 \\ X_1 + X_5 \\ X_2 \\ X_3 \\ 0 \\ 0 \\ X_6 \\ X_7 \end{bmatrix}, \begin{bmatrix} X_0 + X_4 \\ X_1 + X_5 \\ X_2 + X_6 \\ X_3 \\ 0 \\ 0 \\ 0 \\ X_7 \end{bmatrix}, \begin{bmatrix} X_0 + X_4 \\ X_1 + X_5 \\ X_2 + X_6 \\ X_3 + X_7 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\tilde{X} = \begin{bmatrix} X_0 + X_4 \\ X_1 + X_5 \\ X_2 + X_6 \\ X_3 + X_7 \end{bmatrix}$$

IDFT

$$\tilde{\mathbf{X}} = \text{IDFT}(\mathbf{X}) = \tilde{\mathbf{A}}^{-1} \mathbf{X}$$

$$\frac{1}{N} \tilde{\mathbf{A}} \tilde{\mathbf{A}}^* = \mathbf{I}_N \quad \tilde{\mathbf{A}}^{-1} = \frac{1}{N} \tilde{\mathbf{A}}^*$$

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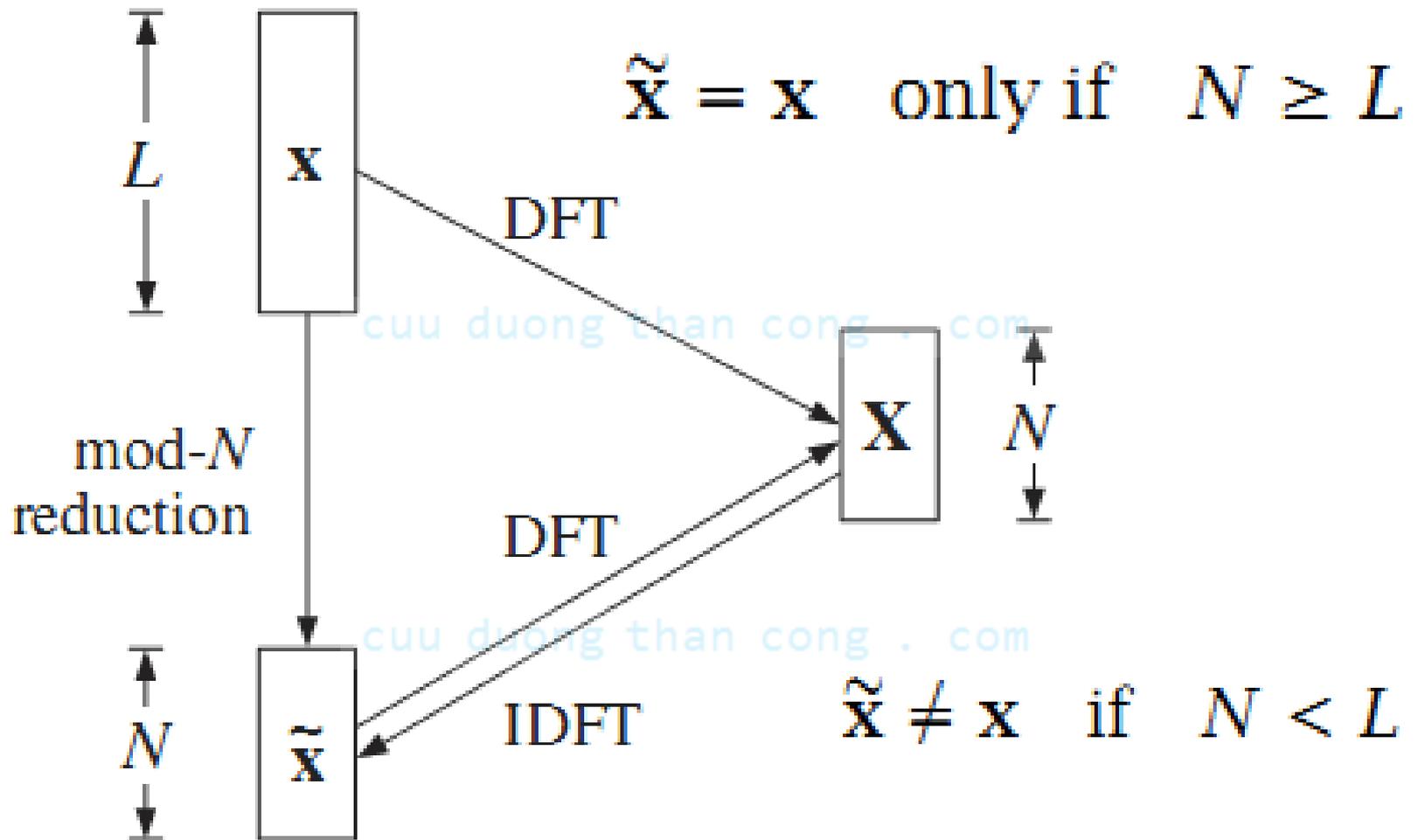
$$\tilde{\mathbf{A}}^* \mathbf{X} = (\tilde{\mathbf{A}} \mathbf{X}^*)^* = [\text{DFT}(\mathbf{X}^*)]^*$$

$$\text{IDFT}(\mathbf{X}) = \frac{1}{N} [\text{DFT}(\mathbf{X}^*)]^*$$

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$$\tilde{x}_n = \frac{1}{N} \sum_{k=0}^{N-1} W_N^{-nk} X_k$$

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(\omega_k) e^{j\omega_k n}$$



Ví dụ

$$\tilde{\mathbf{X}} = \text{IDFT}(\mathbf{X}) = \frac{1}{N} \tilde{\mathbf{A}}^* \mathbf{X} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 6 \\ 8 + 4j \\ -2 \\ 8 - 4j \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$$

$$\frac{1}{N} (\tilde{\mathbf{A}} \mathbf{X}^*)^* = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 6 \\ 8 - 4j \\ -2 \\ 8 + 4j \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ -3 \\ 4 \end{bmatrix}$$

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- Tính trực tiếp DFT N – điểm của x(n):

Tổng quát: X(k) và x(n) là số phức:

$$X_R(k) = \sum_{n=0}^{N-1} \left[x_R(n) \cos \frac{2\pi kn}{N} + x_I(n) \sin \frac{2\pi kn}{N} \right]$$

$$X_I(k) = - \sum_{n=0}^{N-1} \left[x_R(n) \sin \frac{2\pi kn}{N} - x_I(n) \cos \frac{2\pi kn}{N} \right]$$

Tính trực tiếp cần:

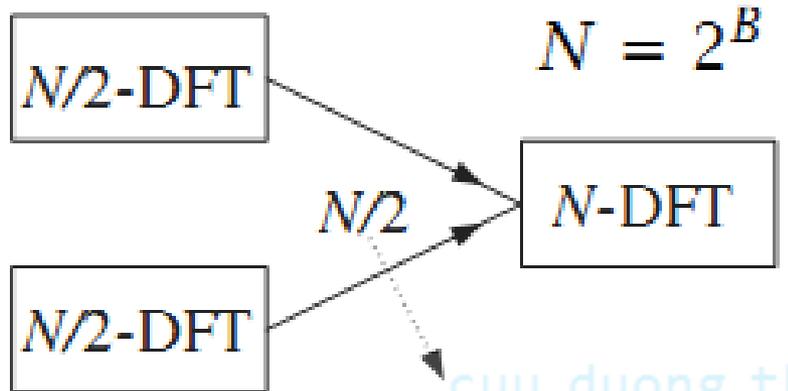
- $2N^2$ phép tính hàm lượng giác
- $4N^2$ phép nhân thực
- $4N(N-1)$ phép cộng thực



Chi phí tính toán lớn

7.3 FFT

- Lịch sử
- Phương pháp chia để trị
- Đánh giá hiệu quả
- FFT phân chia miền thời gian
 - Công thức
 - Sơ đồ hình bướm
- FFT phân chia miền tần số
 - Công thức
 - Sơ đồ hình bướm
- IFFT
 - Quan hệ với FFT



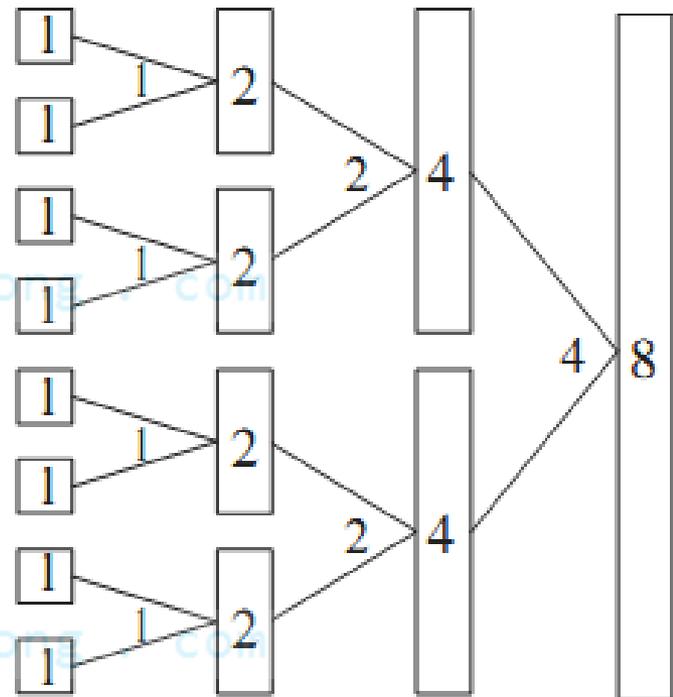
no. of additional multiplications

$(N/2^m)$ -point DFTs

$$\frac{N^2}{2^m} + \frac{N}{2}m$$

$$\frac{1}{2}NB = \frac{1}{2}N \log_2(N)$$

1-DFT 2-DFT 4-DFT 8-DFT



stage 1 stage 2 stage 3

$m = B = \log_2(N)$ stages

$$X(k) = \sum_{n=0}^{N/2-1} W_N^{k(2n)} x(2n) + \sum_{n=0}^{N/2-1} W_N^{k(2n+1)} x(2n+1)$$

$$g(n) = x(2n) \quad n = 0, 1, \dots, \frac{N}{2} - 1 \quad W_M^{k+N/2} = -W_N^k$$

$$h(n) = x(2n+1) \quad W_M^{k+N} = W_N^k$$

$$G(k) = \sum_{n=0}^{N/2-1} W_{N/2}^{kn} g(n) \quad k = 0, 1, \dots, \frac{N}{2} - 1$$

$$H(k) = \sum_{n=0}^{N/2-1} W_{N/2}^{kn} h(n)$$

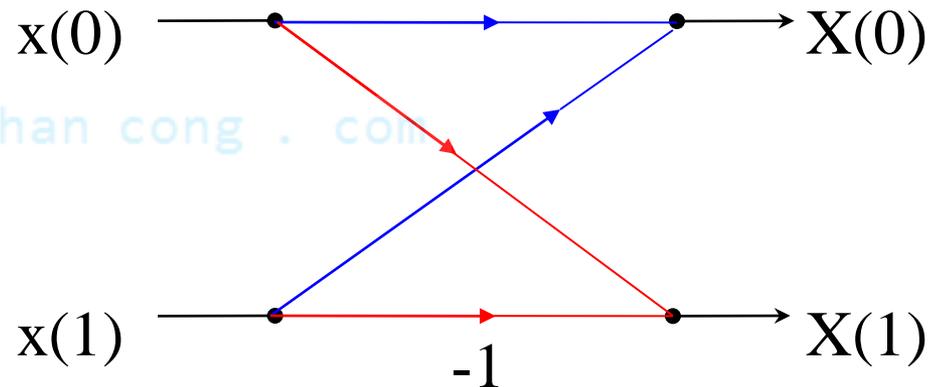
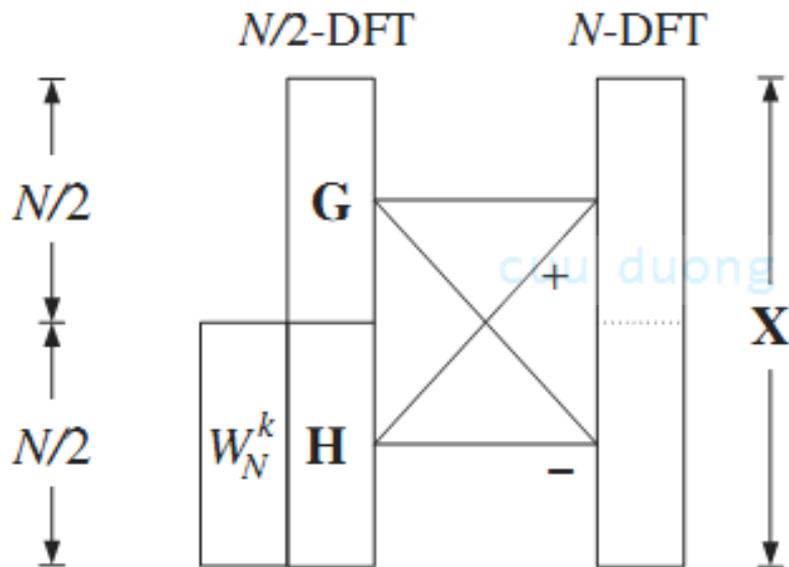
Sơ đồ hình bướm

$$X(k) = G(k) + W_N^k H(k) \quad k = 0, 1, \dots, N - 1$$

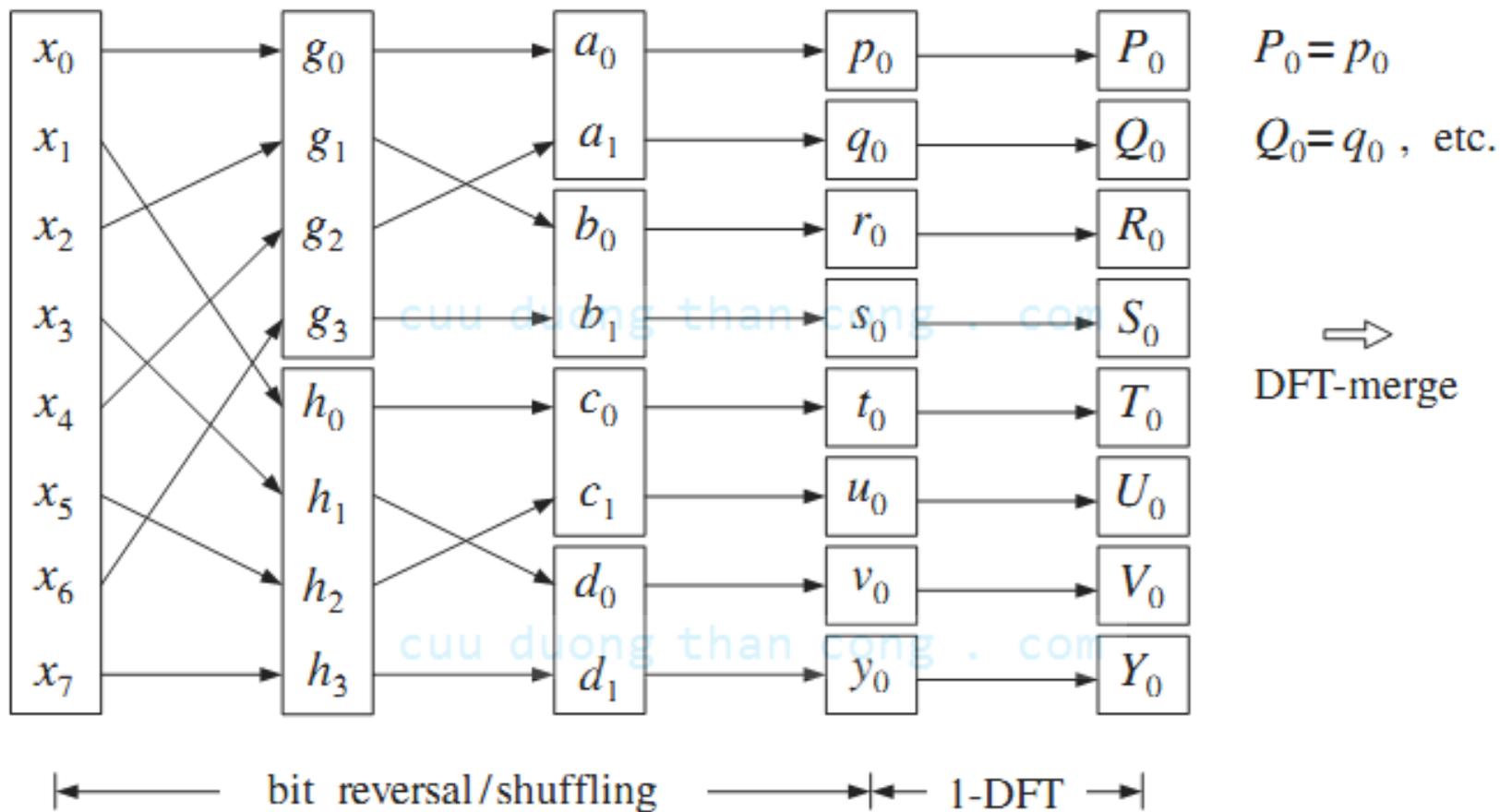
$$X(k) = G(k) + W_N^k H(k)$$

$$k = 0, 1, \dots, \frac{N}{2} - 1$$

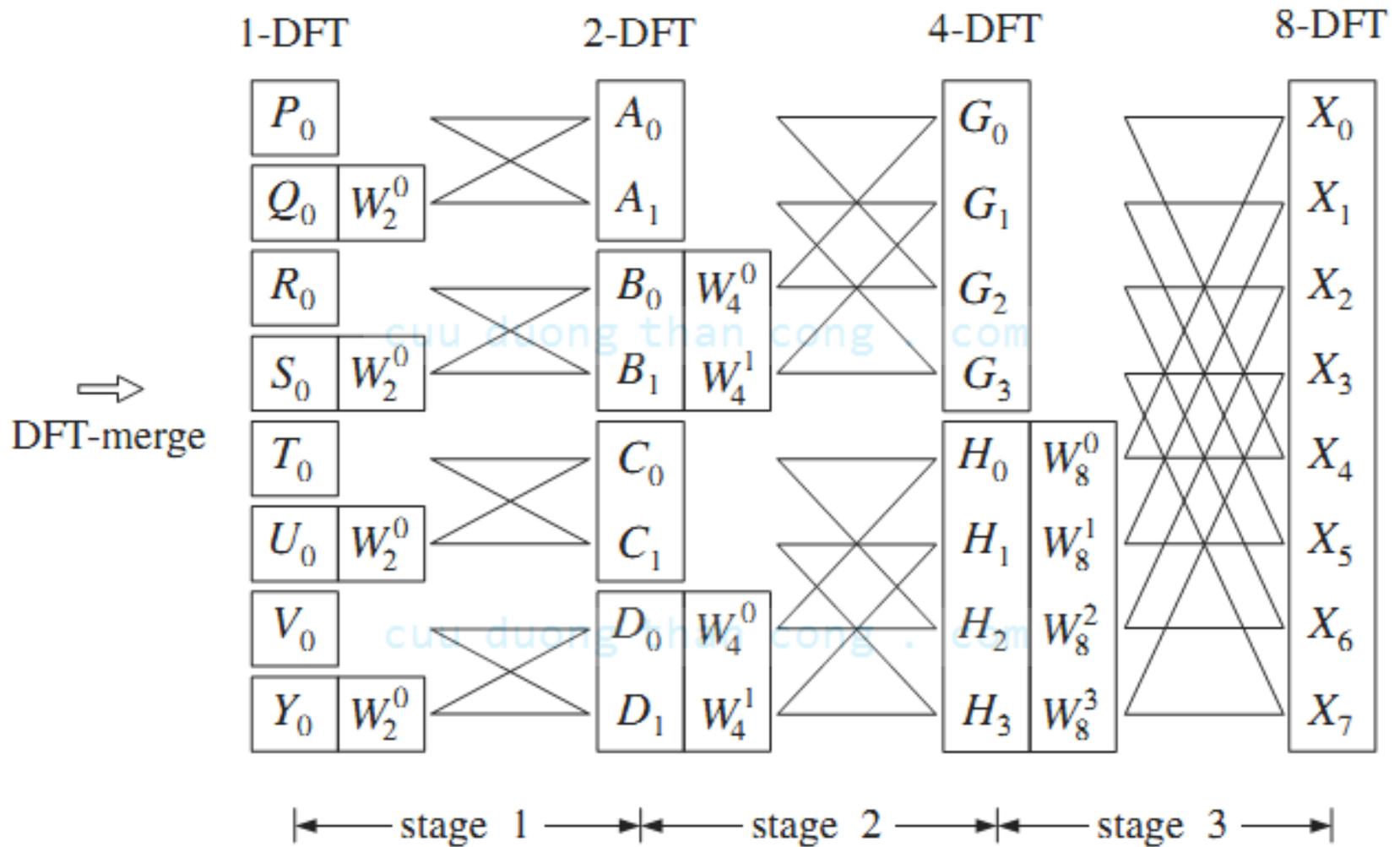
$$X(k + N/2) = G(k) - W_N^k H(k)$$



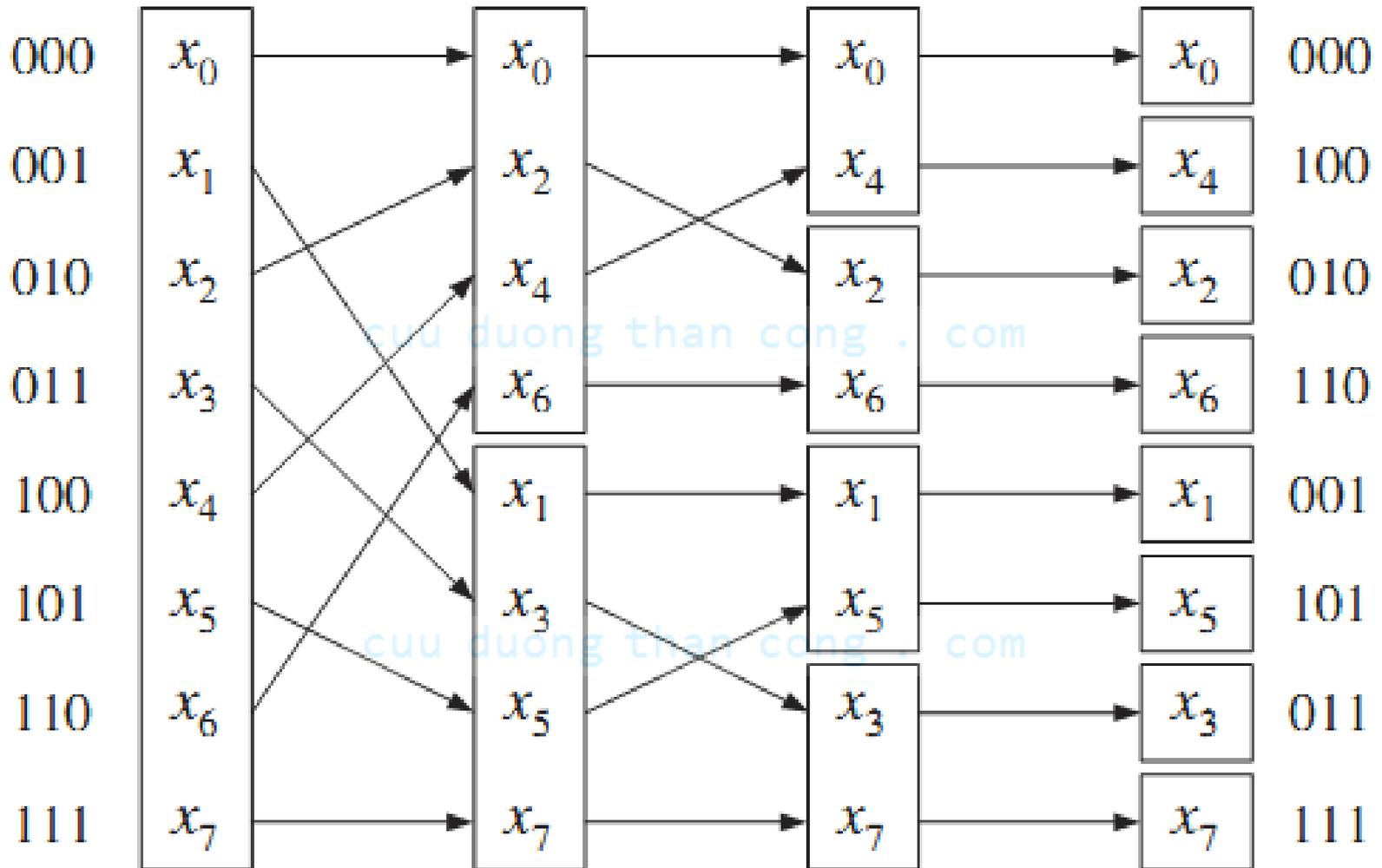
FFT phân chia miền thời gian



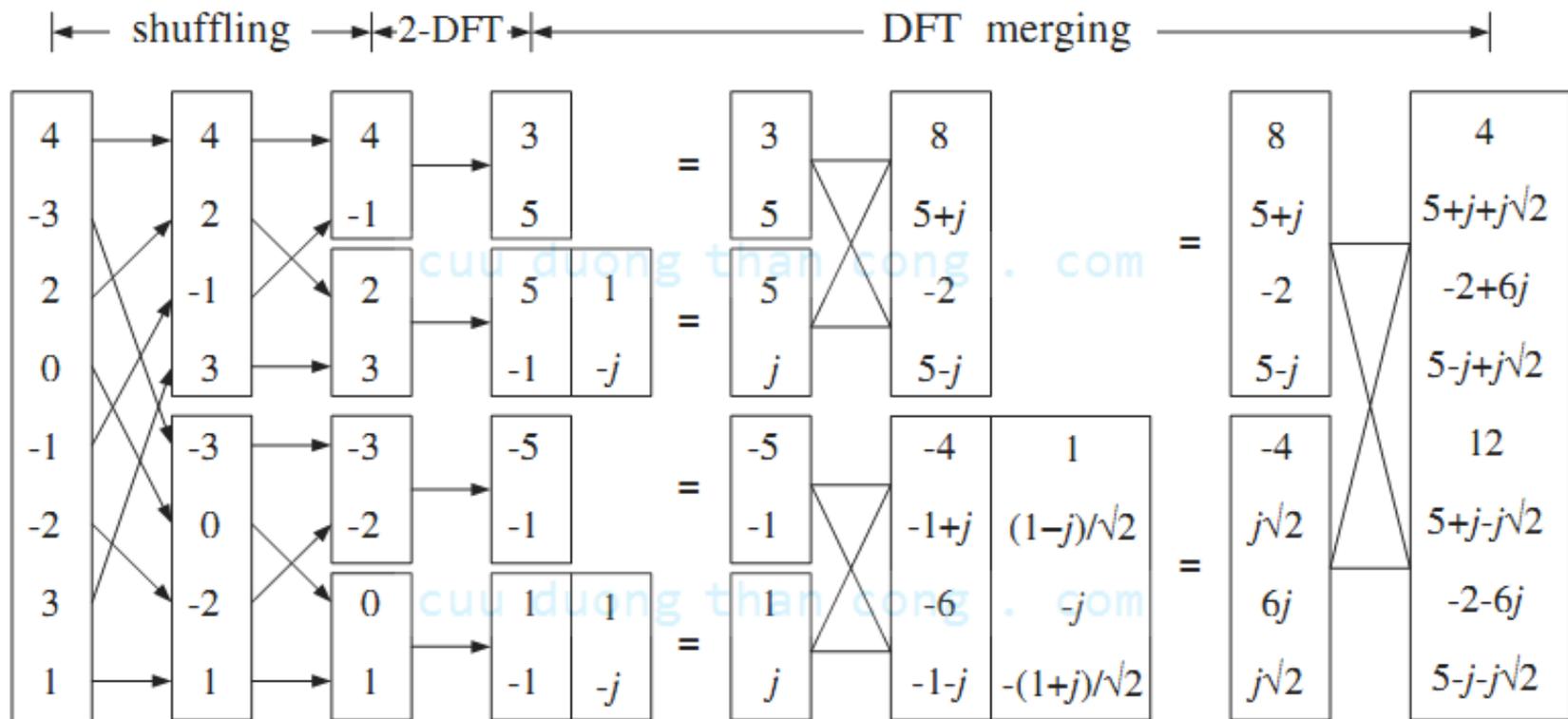
FFT phân chia miền thời gian (2)



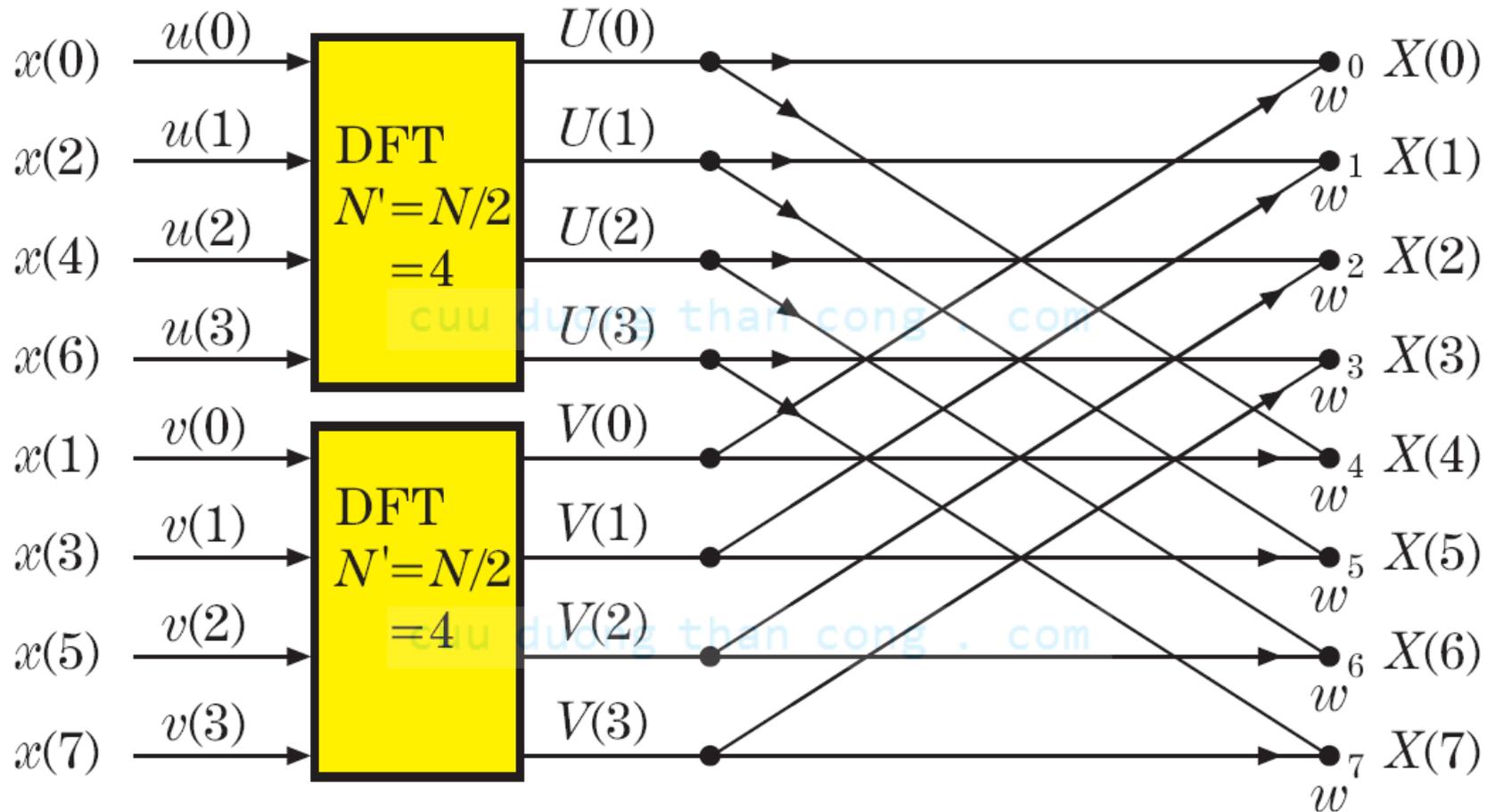
Quá trình đảo bit



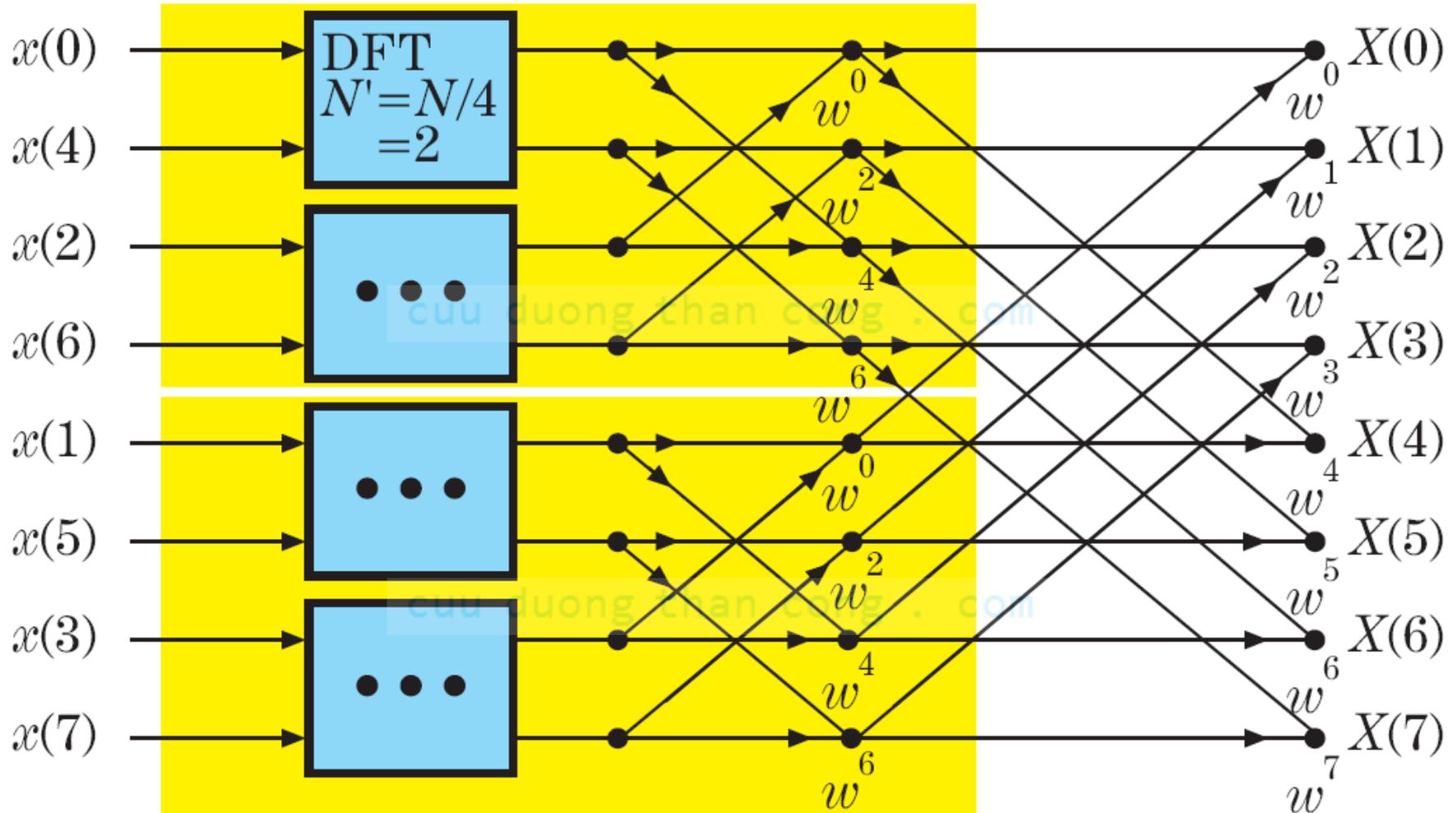
Ví dụ



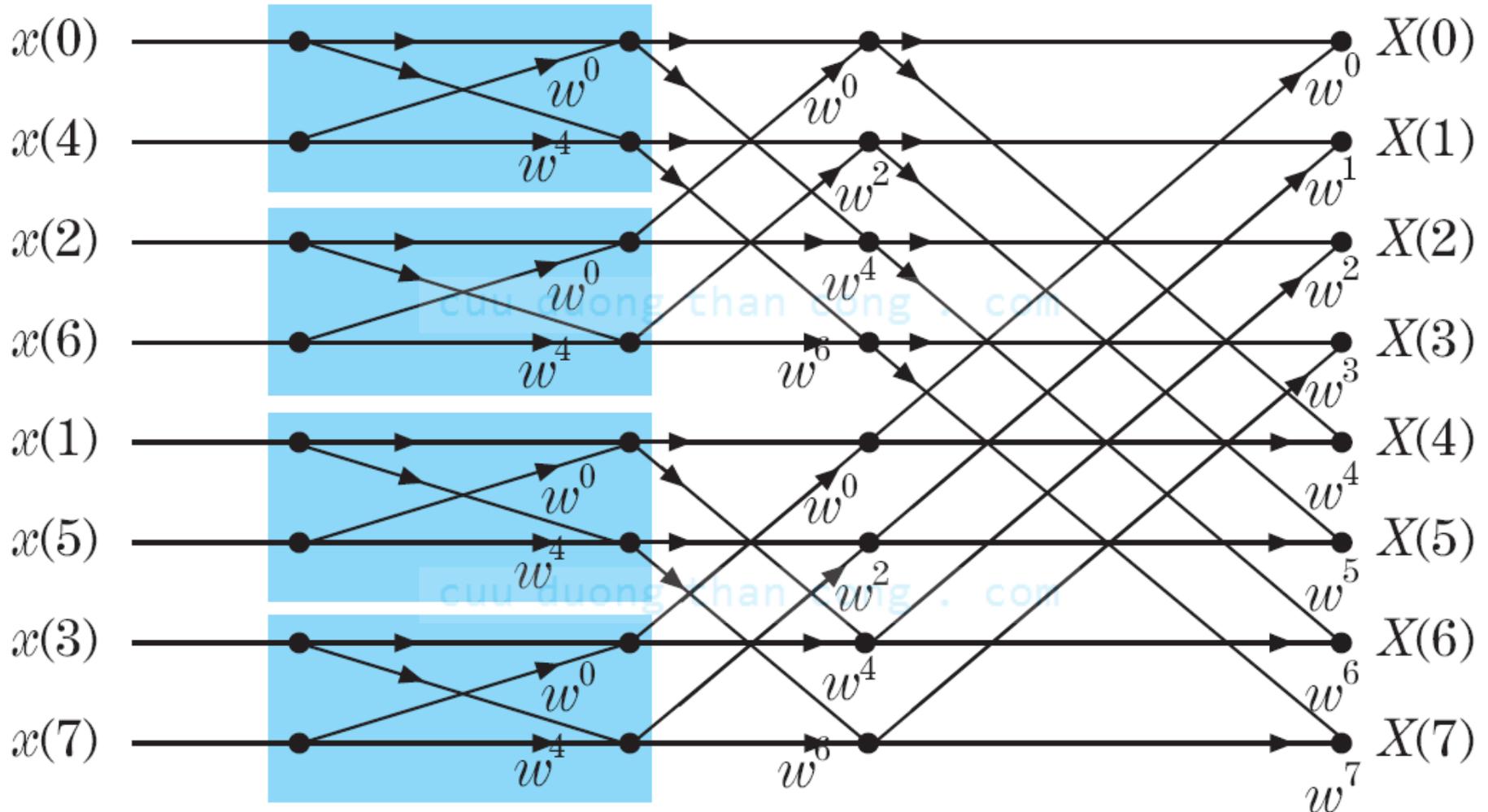
FFT 8 điểm phân chia theo thời gian



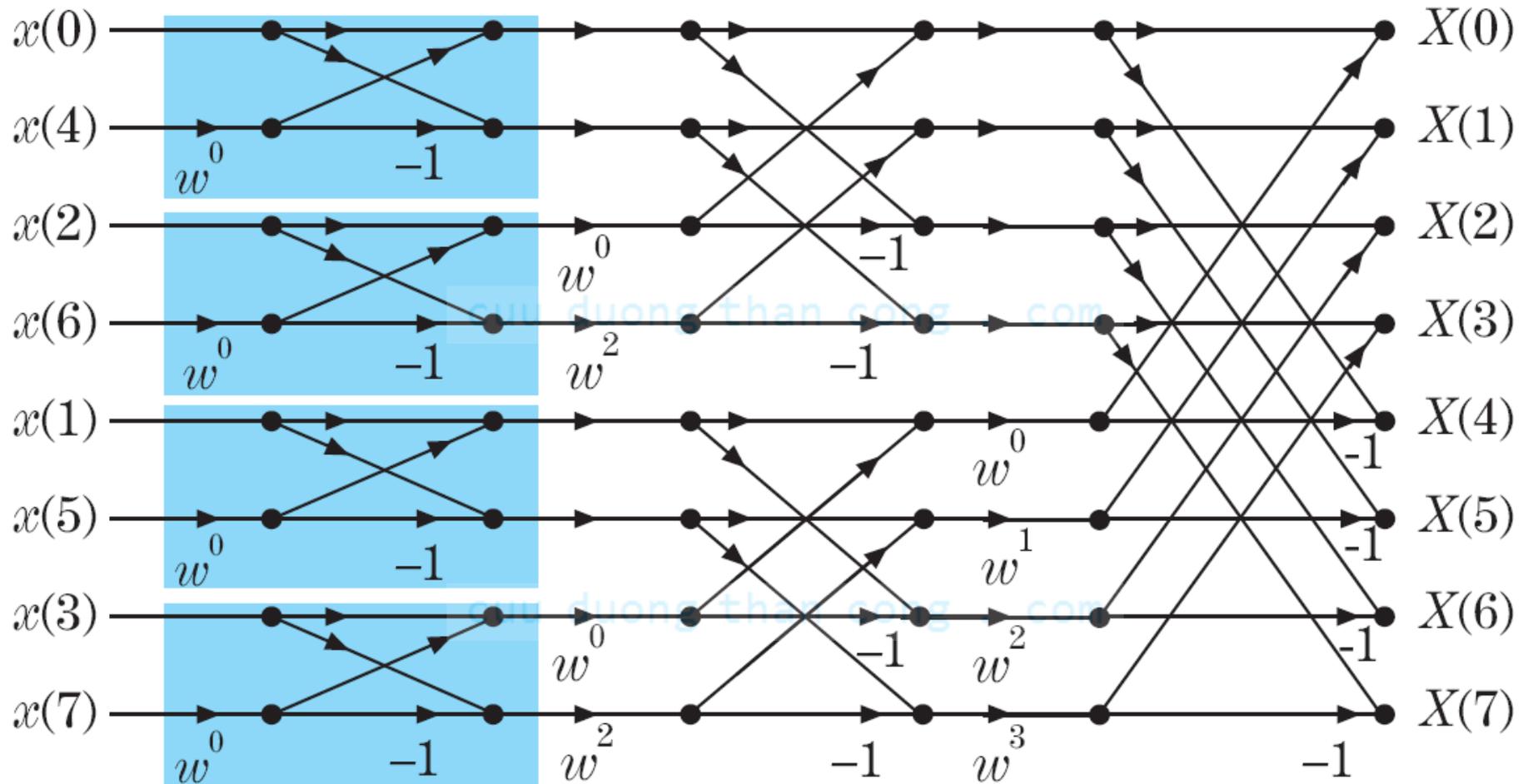
FFT 8 điểm phân chia theo thời gian (2)



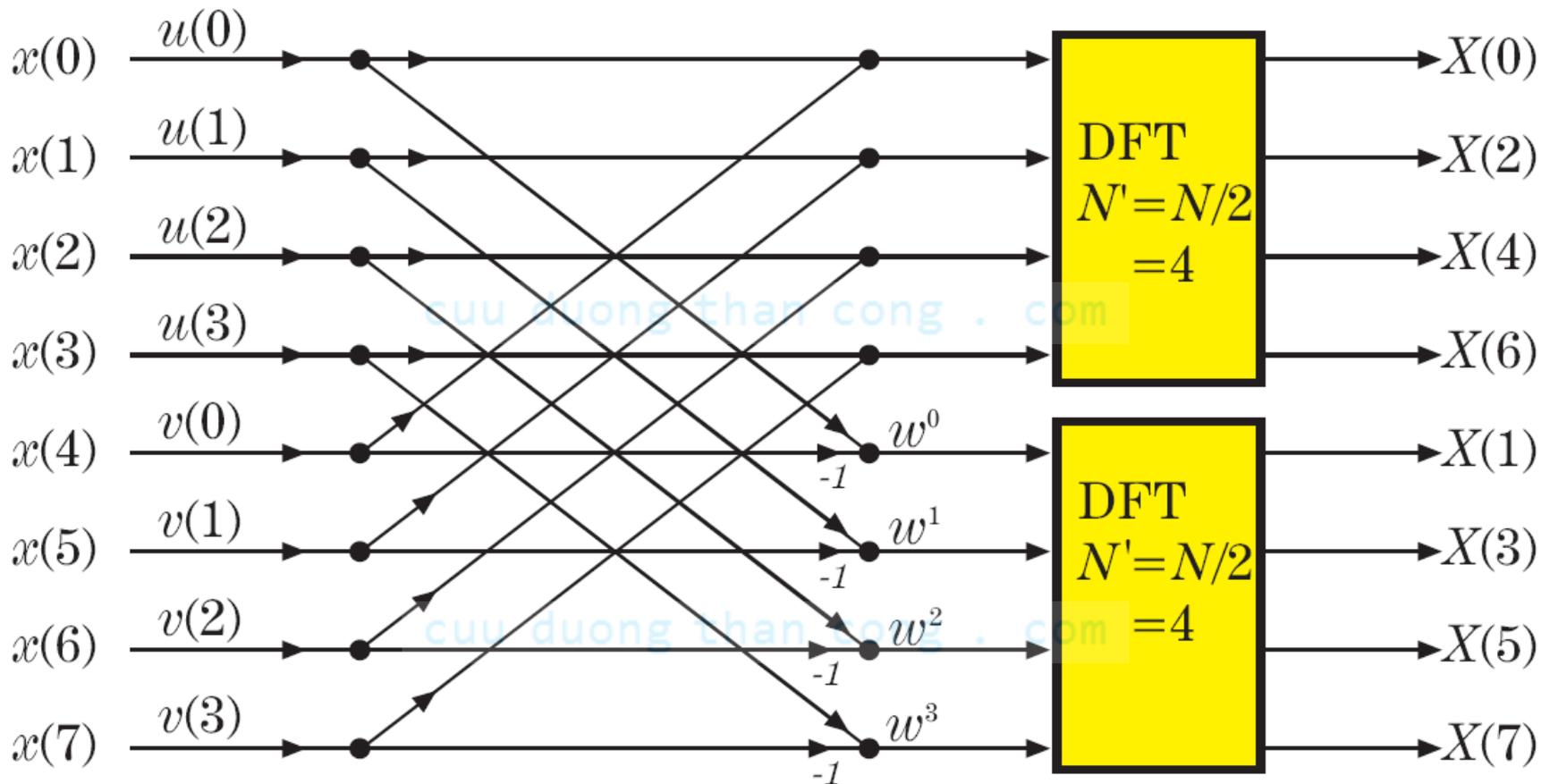
FFT 8 điểm phân chia theo thời gian (3)



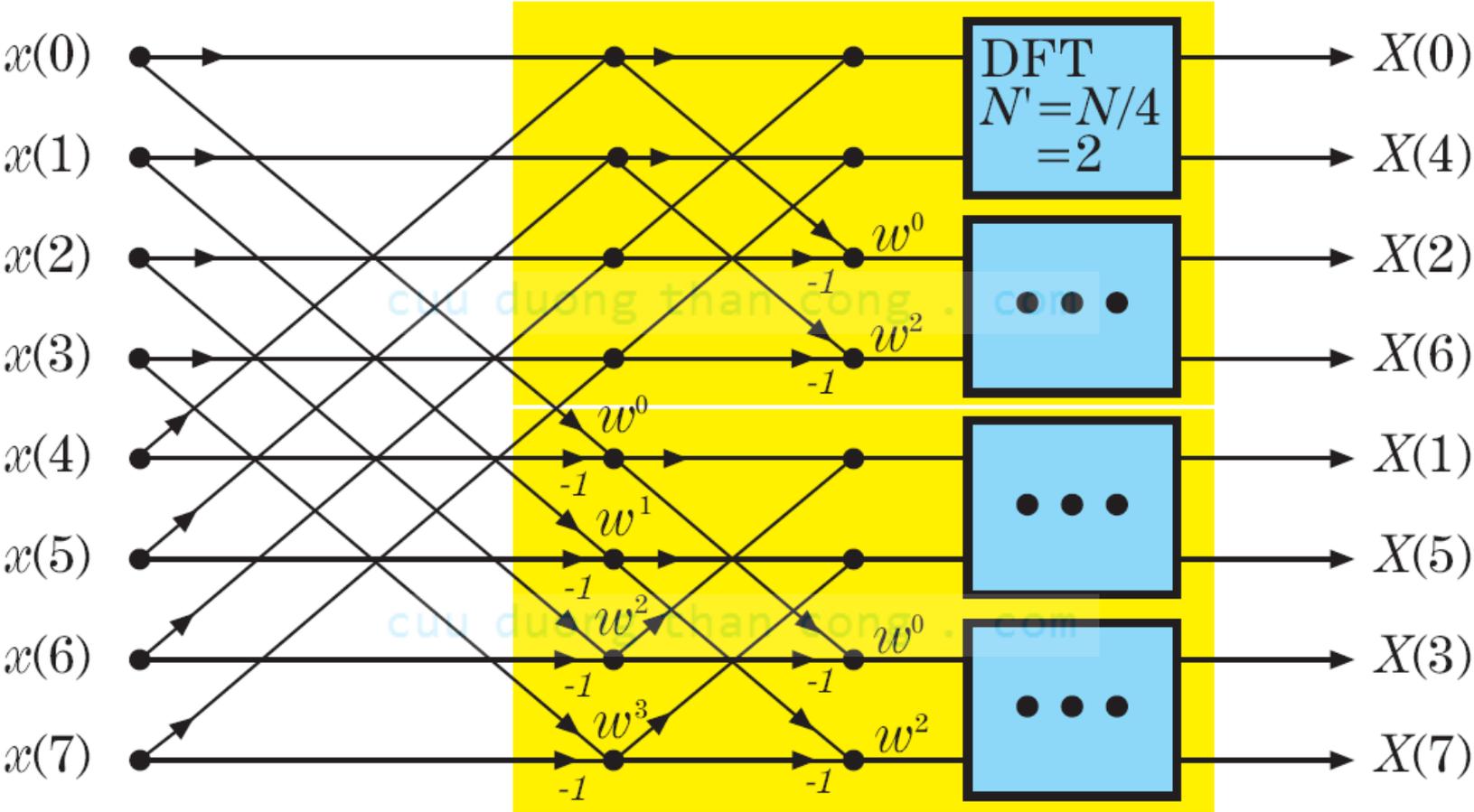
FFT 8 điểm phân chia theo thời gian (4)



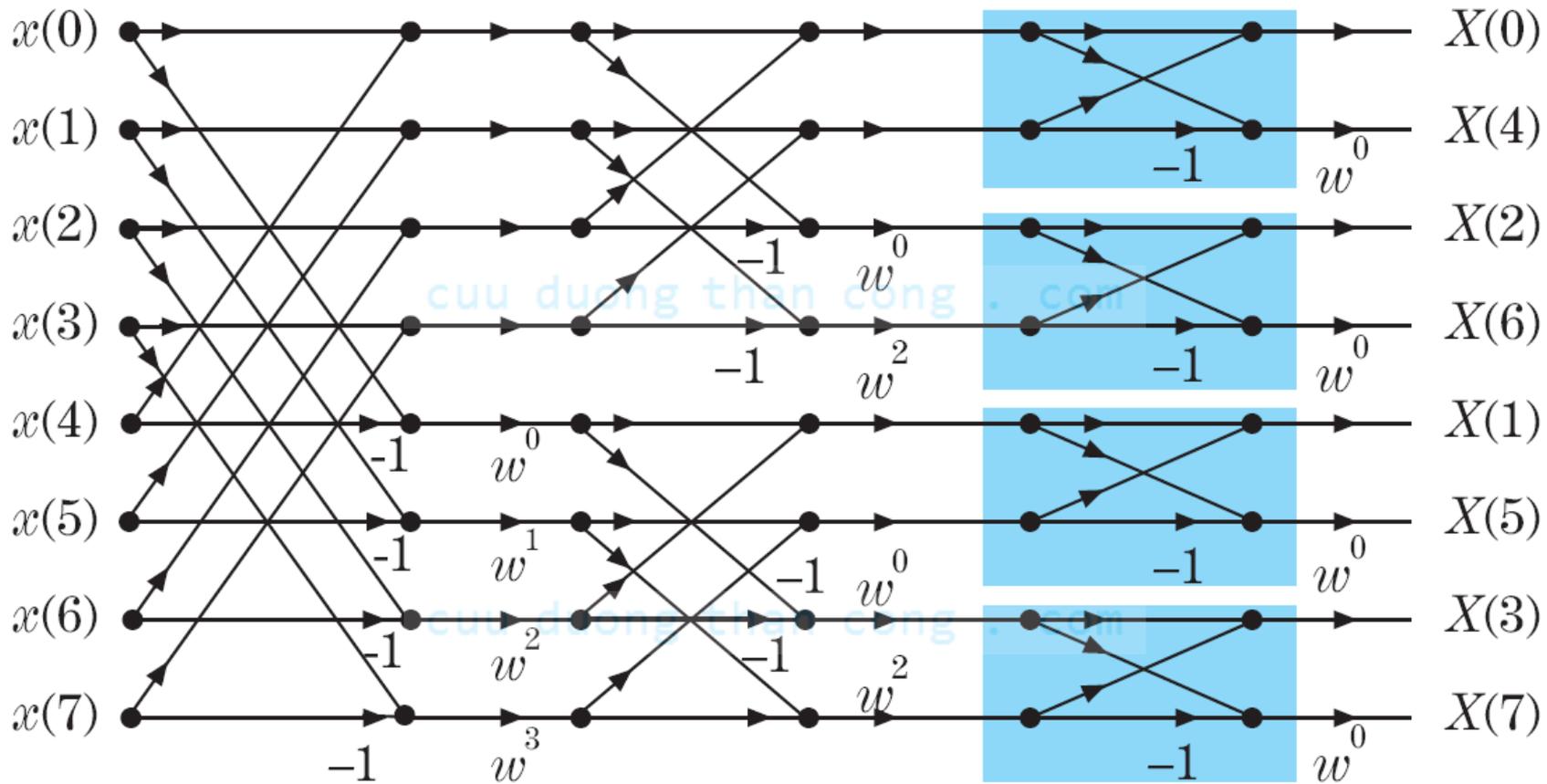
FFT 8 điểm phân chia theo tần số



FFT 8 điểm phân chia theo tần số (2)



FFT 8 điểm phân chia theo tần số (3)



Tóm tắt chương 7

- Ảnh hưởng của quá trình cửa sổ hóa tín hiệu?
- So sánh các hàm cửa sổ?
- Điều kiện của số điểm N khi thực hiện DFT?
- Điều kiện tối ưu của số điểm N khi thực hiện FFT?
- Đánh giá hiệu quả của FFT so với DFT?
- Xác định DFT N điểm?
- Xác định IDFT N điểm?
- Thực hiện FFT N điểm phân chia miền thời gian?
- Thực hiện FFT N điểm phân chia miền tần số?
- Thực hiện IFFT N điểm phân chia miền thời gian?
- Thực hiện IFFT N điểm phân chia miền tần số?