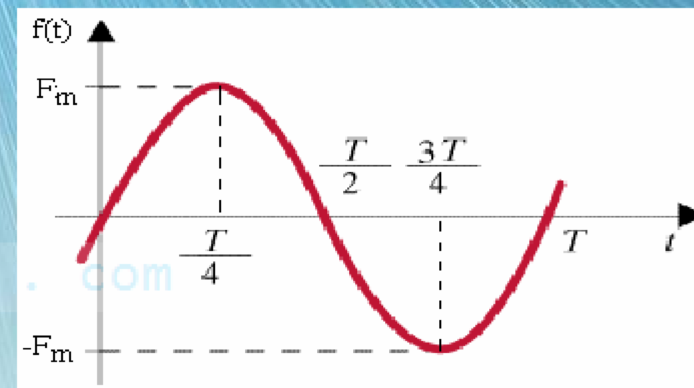
Chapter 1:

Fourier Series

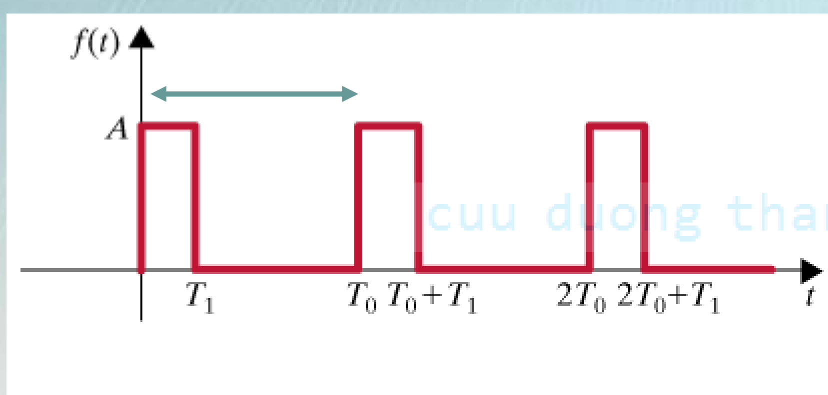
1.1: Periodic Function :

- ❖ **Periodic Function :** $f(t) = f(t + nT)$

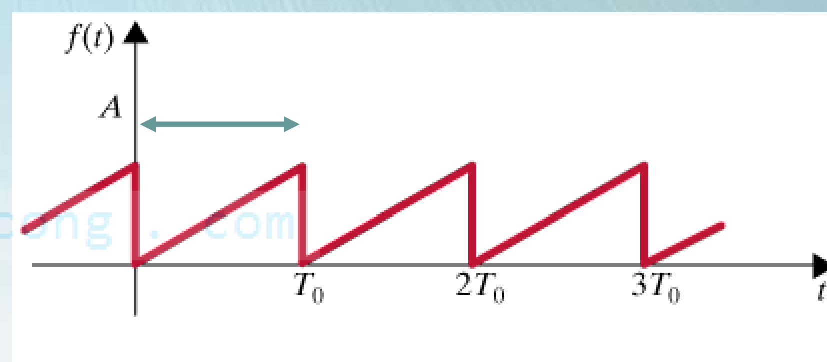
  $(\ n = \pm 1, \pm 2\ \ldots\ ;\ T = \text{const} = \text{period}\ )$



- ❖ **Two types :**
  - **sinwave:**
  - **nonsin wave:**



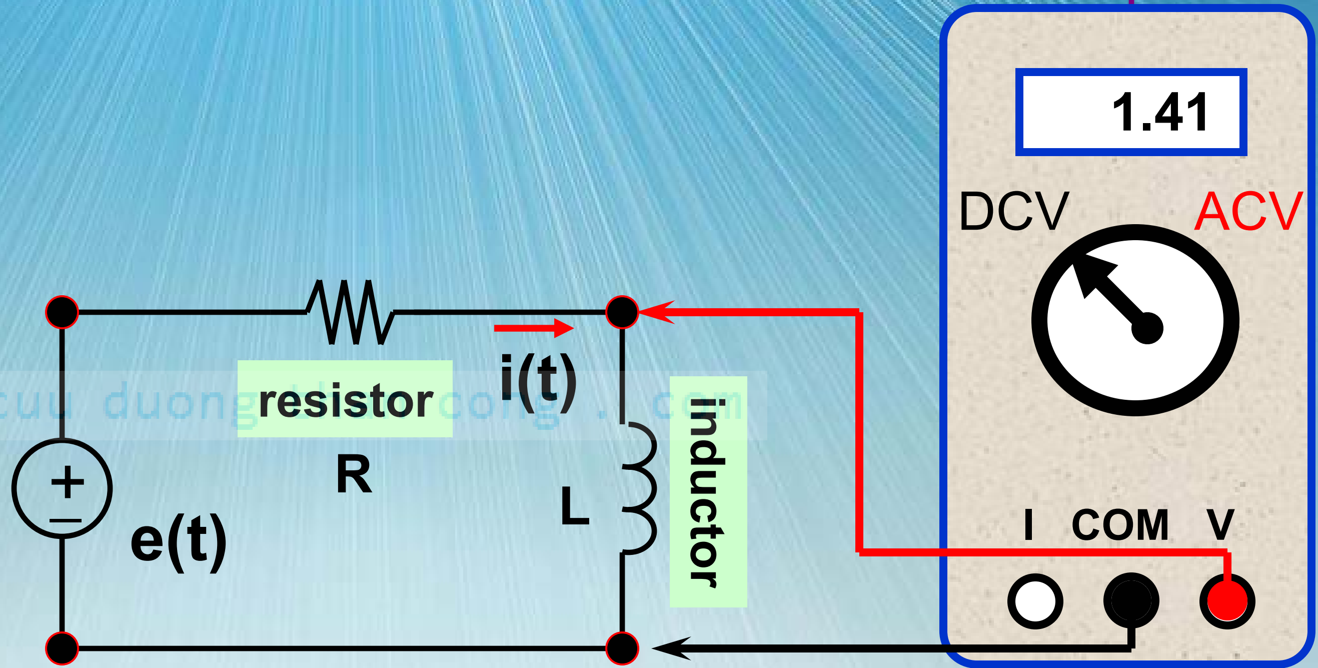
(a)



(c)

Need to research Fourier Seies ?

- **DC analysis:**



- **AC analysis:**

- **Periodic signal analysis:**

1.2: Trigonometric Fourier Series :

1) Trigonometric series:

If f(t) = defined – singlevalued – periodic (Dirichlet Conditions) : can be presented by a Fourier series :

$$f(t) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)\right] \quad (1)$$

Với : n = 1,2 ...

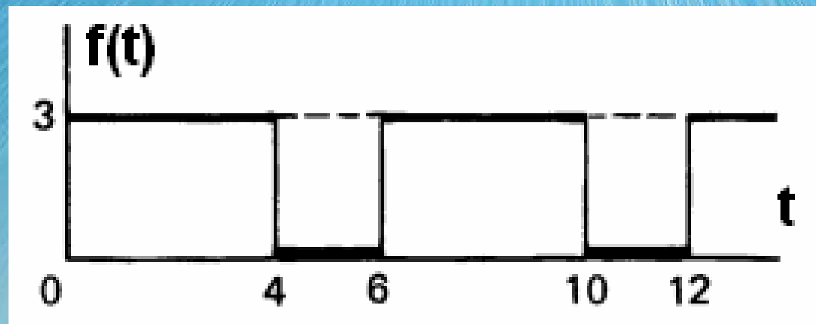
$\omega_0 = 2\pi/T$ = tần số cơ bản

a_0 , a_n , b_n = các hệ số khai triển Fourier .

2) The Fourier Coefficients :

❖ Tín hiệu có chu kỳ T (s)	❖ Tín hiệu có chu kỳ 2π (rad)
$a_0 = \frac{1}{T}\int_0^T f(t)dt$	$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega t)d(\omega t)$
$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)dt$	$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\omega t)\cos(n\omega_0 t)d(\omega t)$
$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)dt$	$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\omega t)\sin(n\omega_0 t)d(\omega t)$

3) Analytic Description of a periodic signal :



a) **Between t = 0 and t = 4: f(t) = 3, i.e f(t) = 3 $0 < t < 4$.**

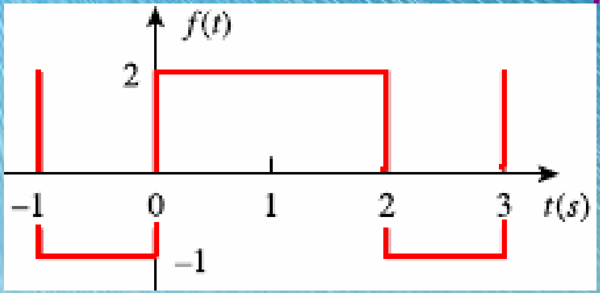
b) **Between t = 4 and t = 6: f(t) = 0, i.e f(t) = 0 $4 < t < 6$.**

⇨ **So we could define the function:**

$$\begin{cases} f(t) = \begin{cases} 3 & 0 < t < 4 \\ 0 & 4 < t < 6 \end{cases} \\ f(t) = f(t+6) \end{cases}$$

❖ Example1: Find Fourier Series ?

a) Determine the Fourier Series ?
b) Checking by MATLAB ?



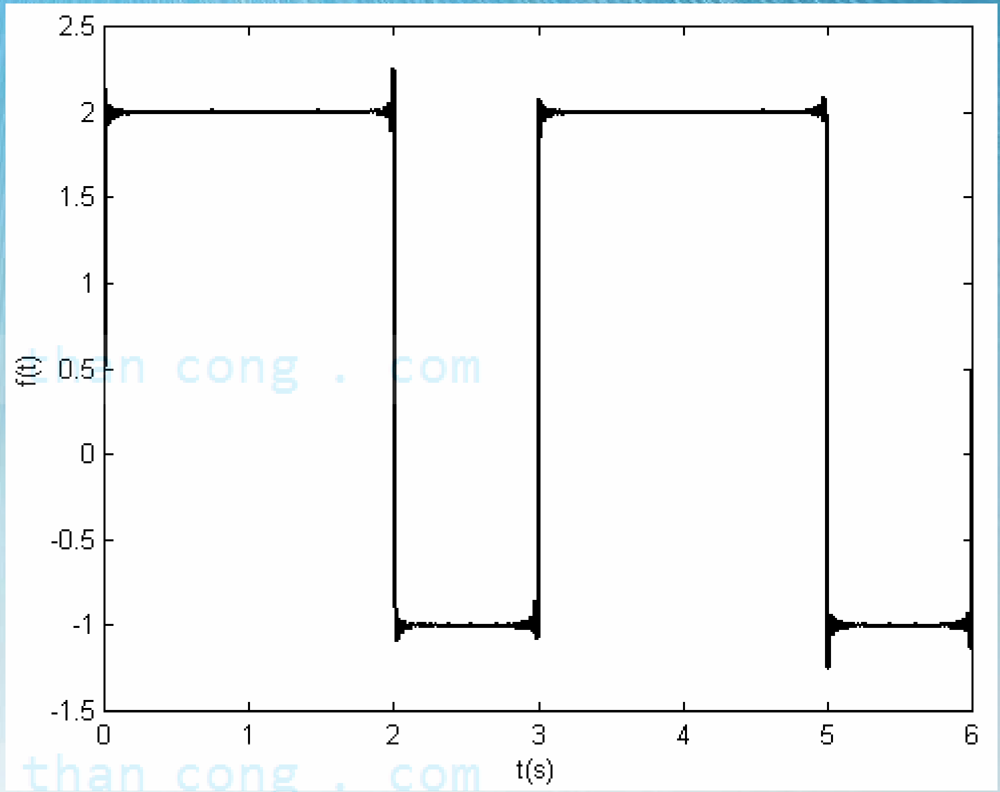
$$T = 3, \qquad \omega_o = 2\pi / T = 2\pi / 3$$

$$a_o = 1 \qquad a_n = \frac{3}{n\pi}\sin\frac{4n\pi}{3} \qquad b_n = \frac{3}{n\pi}\left(1 - \cos\frac{4n\pi}{3}\right)$$

$$\Rightarrow \quad f(t) = 1 + \sum_{n=0}^{\infty}\left[\frac{3}{n\pi}\sin\frac{4n\pi}{3}\cos\frac{2n\pi t}{3} + \frac{3}{n\pi}\left(1 - \cos\frac{4n\pi}{3}\right)\sin\frac{2n\pi t}{3}\right]$$

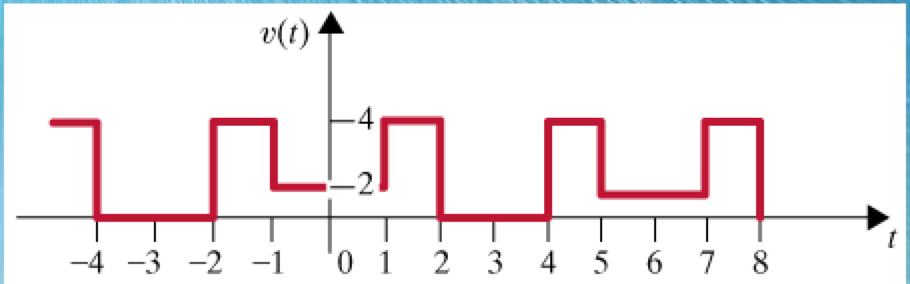
Using MATLAB :

```
pi = 3.14159; N = 100; T = 3; a0 = 1;
w0 = 2*pi/T;
t = linspace(0,2*T,600);
for n=1:N
   a(n)= (3/(n*pi))*sin(4*n*pi/3);
   b(n)= (3/(n*pi))*(1 - cos(4*n*pi/3));
end
for i=1:length(t)
   f(i) = a0;
   for n=1:length(a)
      f(i) = f(i) + a(n)*cos(n*w0*t(i)) +
                     b(n)*sin(n*w0*t(i));
   end
end
plot(t,f,'black');
xlabel('t(s)');
ylabel('f(t)');
```



1.3: Symmetrycal Functions :

a) **Hàm chẵn f(t) = f(-t) : Tín hiệu nhận trục tung làm trục đối xứng.**



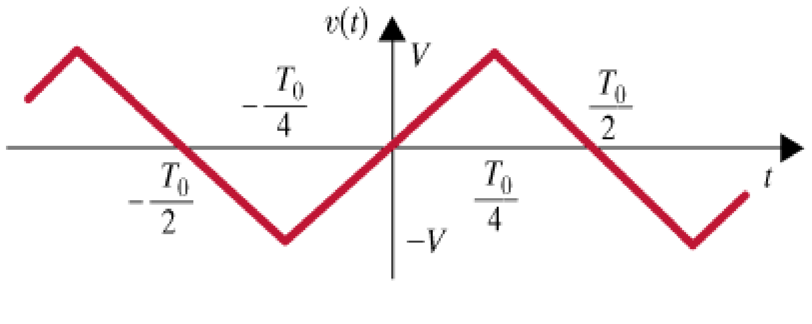
$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)\,dt$$

$$b_n = 0$$

b) Odd Function :

Hàm lẻ f(t) = - f(-t) : Tín hiệu nhận gốc tọa độ làm tâm đối xứng.



(a)

$$\boxed{a_0 = 0}$$

$$\boxed{a_n = 0}$$

$$\boxed{b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt}$$

c) Half-wave Function :

❖ **Hàm đối xứng nửa sóng :**

f(t) = - f(t ± T/2) :

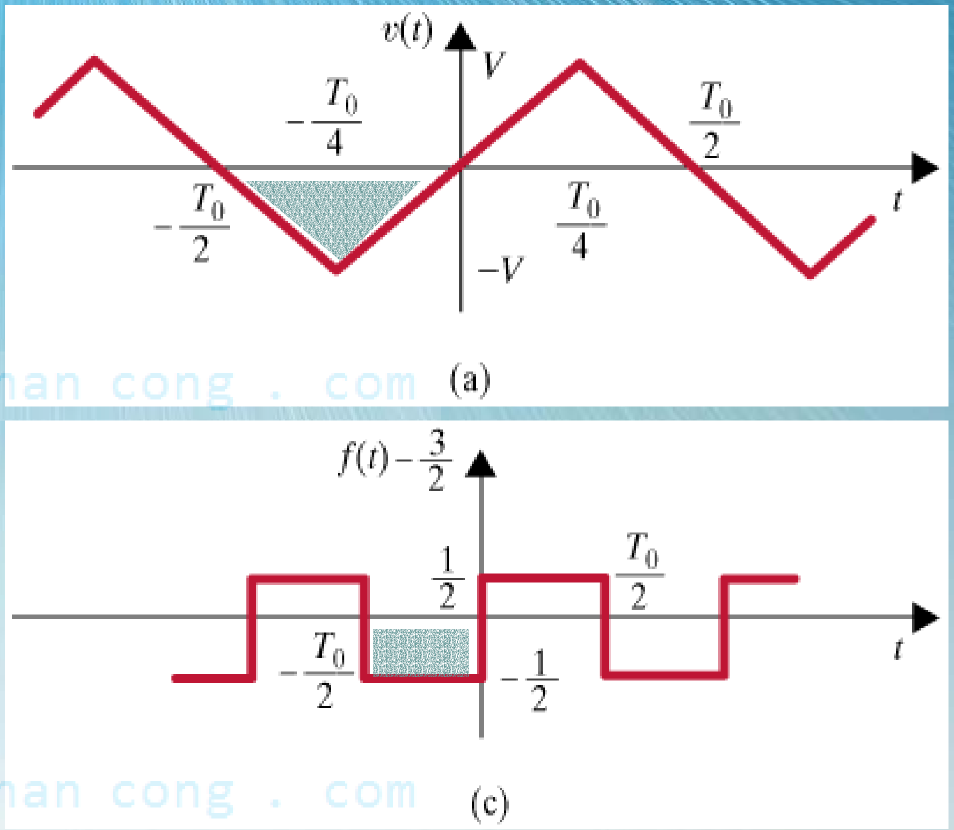
✓ **TP DC :** $\boxed{a_0 = 0}$

✓ **Khi n chẵn :** $\boxed{a_n = b_n = 0}$

✓ **Khi lẻ :**

$$\boxed{a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos(n\omega_0 t)dt}$$

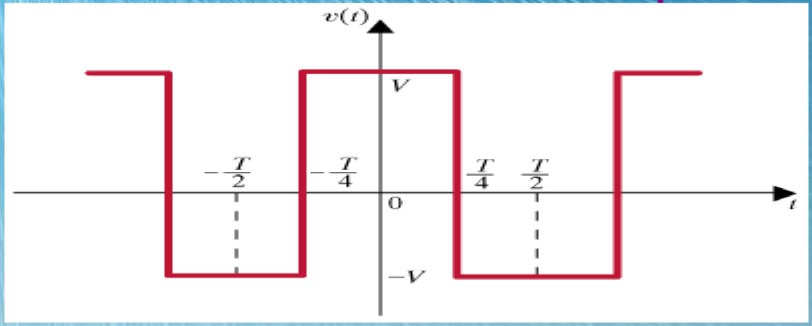
$$\boxed{b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin(n\omega_0 t)dt}$$



Examples of signals with half-wave symmetry

d) Half-wave and even Function :

even $\Rightarrow b_n = 0,\ n = 1,2,\ldots$



half-wave symmetry $\Rightarrow a_n = \dfrac{4}{T_0}\displaystyle\int_0^{\frac{T_0}{2}} f(t)\cos n\omega_0 t\,dt$ for n odd

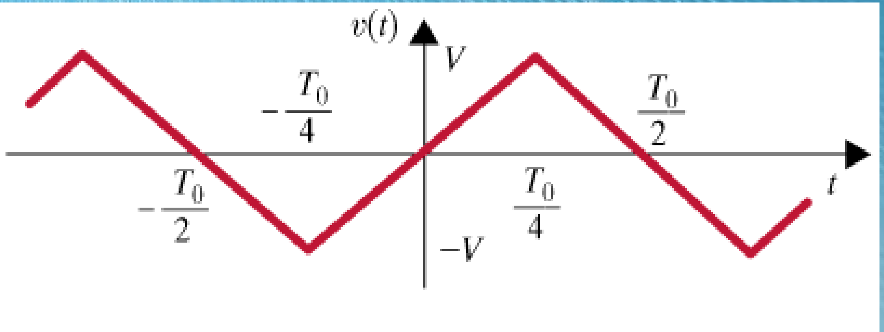
$a_n = 0$ for n even

$$a_{2k+1} = \frac{4}{T}\left[\int_0^{\frac{T}{4}} V\cos(2k+1)\omega_0 t\,dt - \int_{\frac{T}{4}}^{\frac{T}{2}} V\cos(2k+1)\omega_0 t\,dt\right] = \frac{8}{T}\int_0^{\frac{T}{4}} V\cos(2k+1)\omega_0 t\,dt$$

$$a_n = \frac{8}{T}\int_0^{T/4} f(t)\cos(n\omega_0 t)\,dt \quad \text{(n:odd)}$$

e) Half-wave and odd Function :

odd $\Rightarrow \; a_n = 0; \; n = 0,1,2,\ldots$



(a)

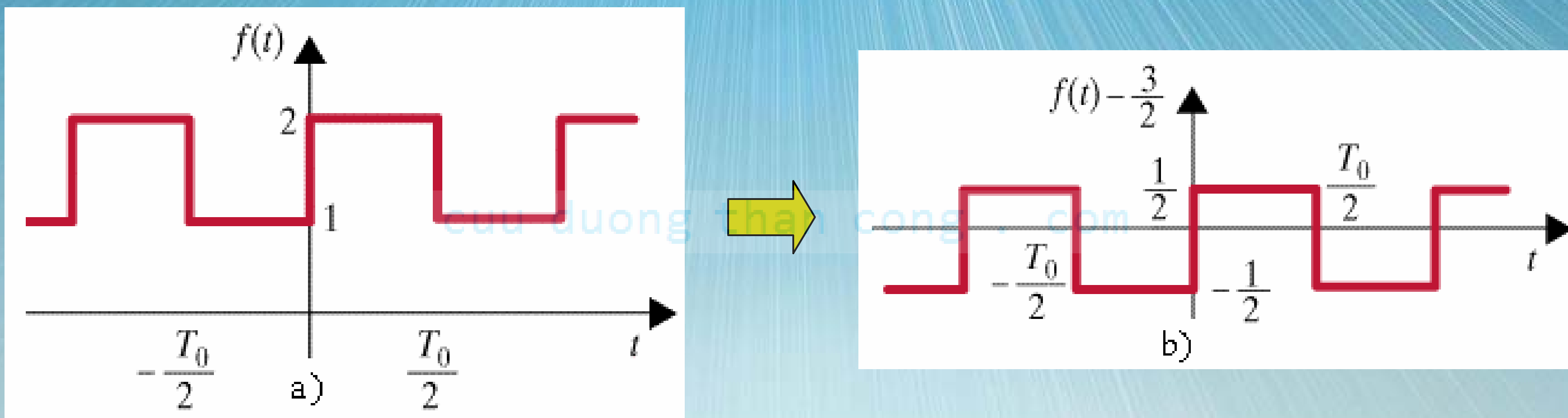
half - wave symmetry $\Rightarrow b_n = \frac{4}{T_0}\int_0^{\frac{T_0}{2}} f(t)\sin n\omega_0 t\, dt$ for n odd

$b_n = 0$ for n odd

$$b_n = \frac{8}{T}\int_0^{T/4} f(t)\sin(n\omega_0 t)dt \quad \text{(n:odd)}$$

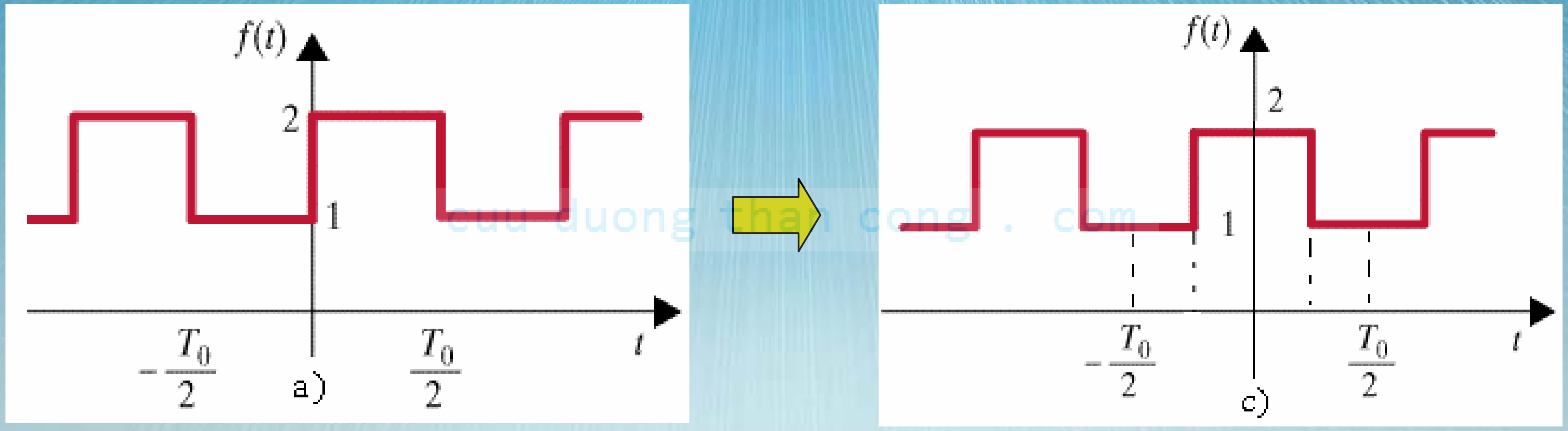
f) With no symmetry :

f_1) DC level Changing:



f) With no symmetry :

f_2) <u>Time Shift:</u>



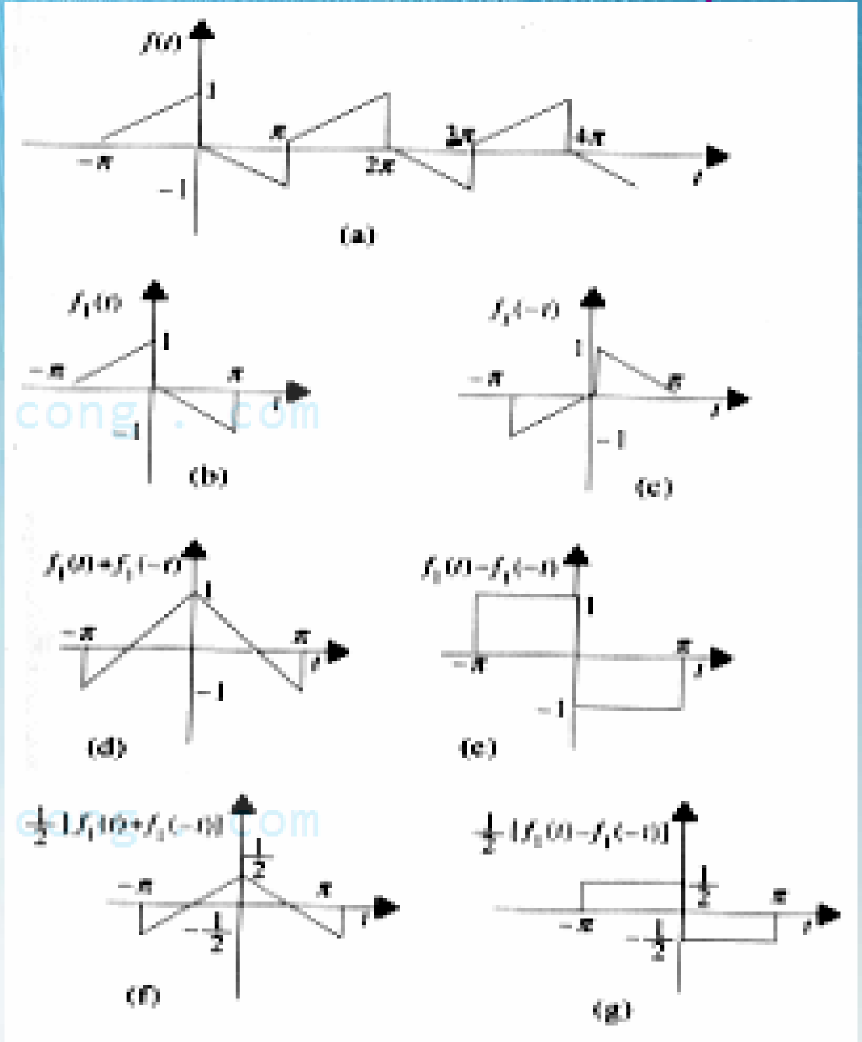
f) With no symmetry :

f_3) Resolve into two components :

$$f_e(t) = \frac{f(t) + f(-t)}{2}$$

$$f_o(t) = \frac{f(t) - f(-t)}{2}$$

[Hàm f(-t) xác định bằng đồ thị]



❖ Và ta có :

$a_0 = a_{0e}$ $a_n = a_{ne}$ $b_n = b_{no}$

1.4: The Alternative Forms of Fourier Series:

1) Cosine Fourier Series:

$$f(t) = d_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \varphi_n) \qquad (2)$$

- **Với :**

d_0 = thành phần DC (trung bình).

$D_1\cos(\omega_0 t + \varphi_1)$ = Tp hài cơ bản.

$D_k\cos(k\omega_0 t + \varphi_k)$ = Tp hài thứ k.

$$\begin{cases} d_0 = a_0 \\ D_n = \sqrt{a_n^2 + b_n^2} \\ \varphi_n = -arctg\dfrac{b_n}{a_n} \end{cases}$$

2) Exponential Fourier Series :

❖ **Nếu sử dụng các công thức biến đổi Euler vào phương trình (1) , ta nhận được chuỗi Fourier dạng số mũ (dạng số phức) như sau :**

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \qquad (3)$$

➢ **Với số phức C_n :**

$$C_n = \frac{1}{T}\int_0^T f(t).e^{-jn\omega_0 t}dt$$

➢ **Và :**

$$C_0 = \frac{1}{T}\int_0^T f(t)dt = a_0 = d_0$$

Chuỗi dạng mũ và chuỗi lượng giác:

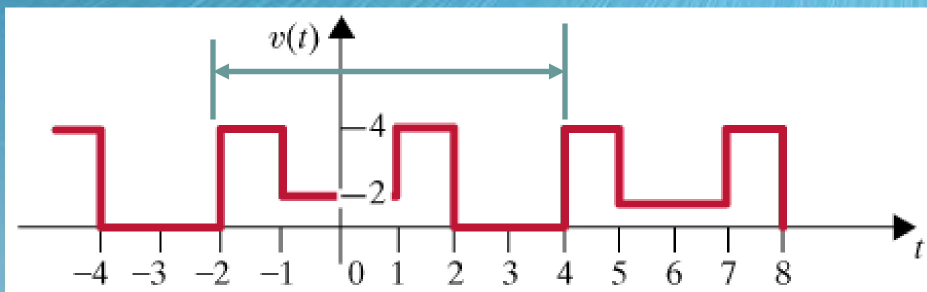
- **Chuỗi dạng mũ quan hệ với các dạng khác :**

$$C_n = \frac{a_n - jb_n}{2} = \frac{\sqrt{a_n^2 + b_n^2}}{2} \angle -arctg\left(\frac{b_n}{a_n}\right) = \frac{D_n}{2} \angle \varphi_n$$

- **Và như vậy :** $$f(t) = C_0 + \sum_{n=1}^{\infty} 2\left|C_n\right| \cos(n\omega_0 t + \angle C_n) \quad (4)$$

❖Ví dụ 1: Chuỗi Fourier mũ.

Determine the exponential Fourier series



$T_0 = 6 \qquad \omega_0 = \dfrac{\pi}{3}$

$$C_0 = \frac{1}{6}\int_{-2}^{2} v(t)dt = 2$$

$$C_n = \frac{1}{6}\int_{-2}^{4} v(t).e^{-jn\omega_0 t}dt = \frac{1}{6}\left[\int_{-2}^{-1} 4e^{-j\frac{n\pi}{3}t}dt + \int_{-1}^{1} 2e^{-j\frac{n\pi}{3}t}dt + \int_{1}^{2} 4e^{-j\frac{n\pi}{3}t}dt\right]$$

$$C_n = \frac{1}{j2n\pi}\left[4e^{j\frac{2n\pi}{3}} - 4e^{j\frac{n\pi}{3}} + 2e^{j\frac{n\pi}{3}} - 2e^{-j\frac{n\pi}{3}} + 4e^{-j\frac{n\pi}{3}} - 4e^{-j\frac{2n\pi}{3}}\right]$$

$$\Rightarrow \quad C_n = \frac{1}{n\pi}\left[4\sin\frac{2n\pi}{3} - 2\sin\frac{n\pi}{3}\right]$$

1.5: Applications of Fourier Series:

1) Frequency Spectra:

$$f(t) = d_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \varphi_n) \quad (2)$$

Cosine expansion

- Phổ biên độ : biểu diễn D_n theo n .
- Phổ pha : biểu diễn φ_n theo n .

❖ Example1: Frequency Spectra

Determine and plot the first four terms of the spectrum

$$v(t) = \sum_{\substack{n=1 \\ n\ odd}}^{\infty} \left(\frac{20}{n\pi}\sin n\omega_0 t - \frac{40}{n^2\pi^2}\cos n\omega_0 t\right)$$

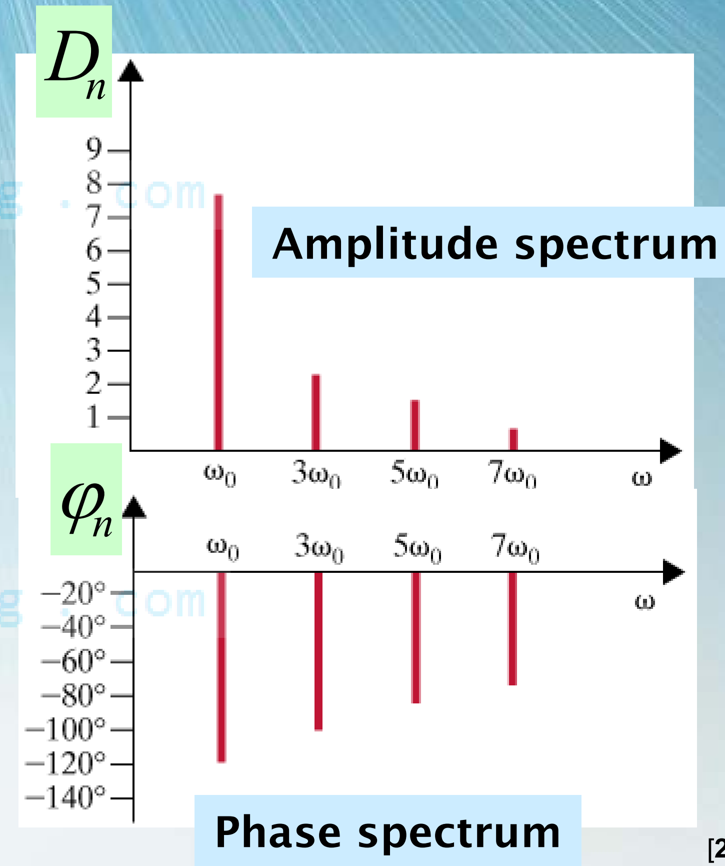
$$\boxed{D_n\angle\varphi_n = 2C_n = a_n - jb_n}$$

$$D_1\angle\varphi_1 = -\frac{40}{\pi^2} - j\frac{20}{\pi} = 7.5\angle -122^\circ$$

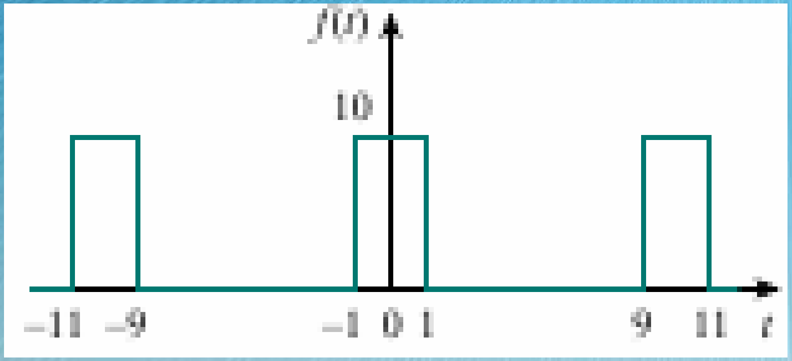
$$D_3\angle\varphi_3 = -\frac{40}{9\pi^2} - j\frac{20}{3\pi} = 2.2\angle -102^\circ$$

$$D_5\angle\varphi_5 = 1.3\angle -97^\circ$$

$$D_7\angle\varphi_7 = 0.92\angle -95^\circ$$



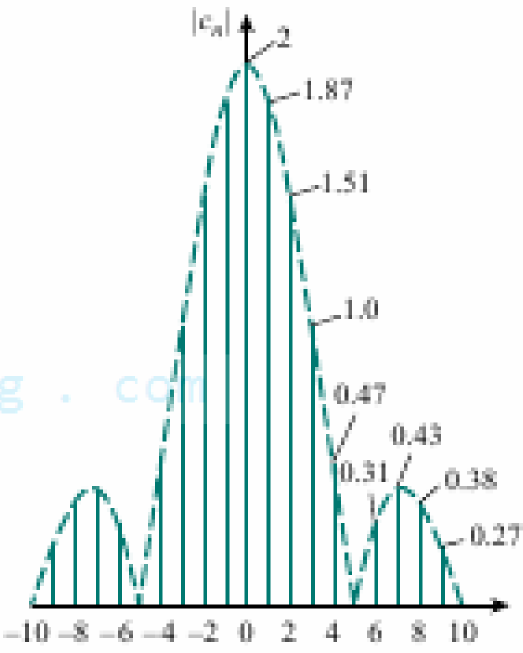
❖ Example2: Frequency Spectra ?



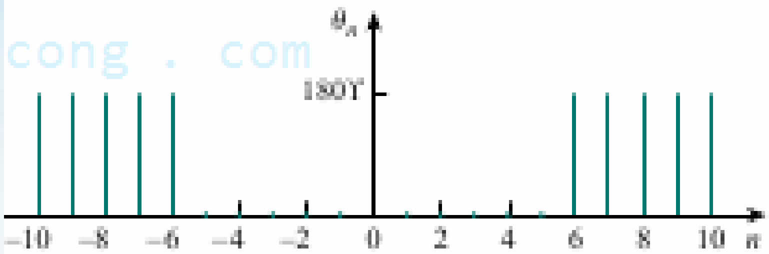
$$C_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)e^{-jn\omega_0 t}\,dt$$

$$|C_n| = 2\left|\frac{\sin(n\pi/5)}{n\pi/5}\right|$$

$$\varphi_n = \begin{cases} 0, & \sin(n\pi/5) > 0 \\ 180^0, & \sin(n\pi/5) < 0 \end{cases}$$



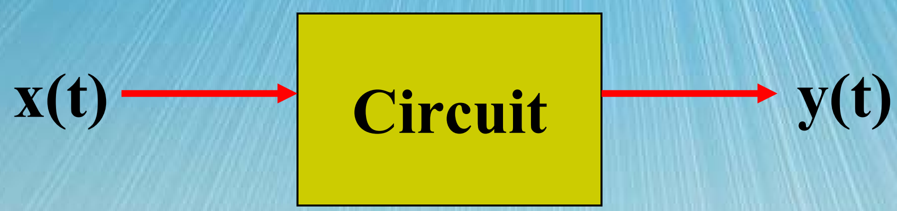
The amplitude of a periodic pulse train.



The phase spectrum of a periodic pulse train.

2) Response to Periodic Excitations:

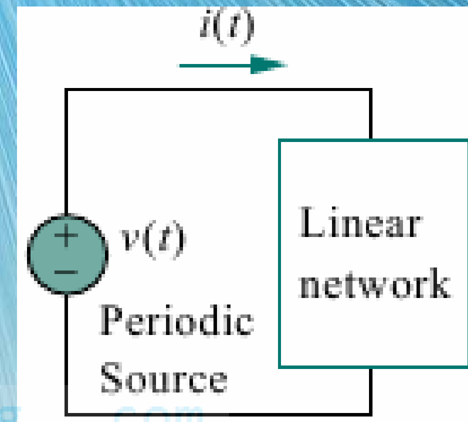
a) Problem:



Find the steady-state response y(t) if x(t) = periodic signal ?

b) General Procedure:

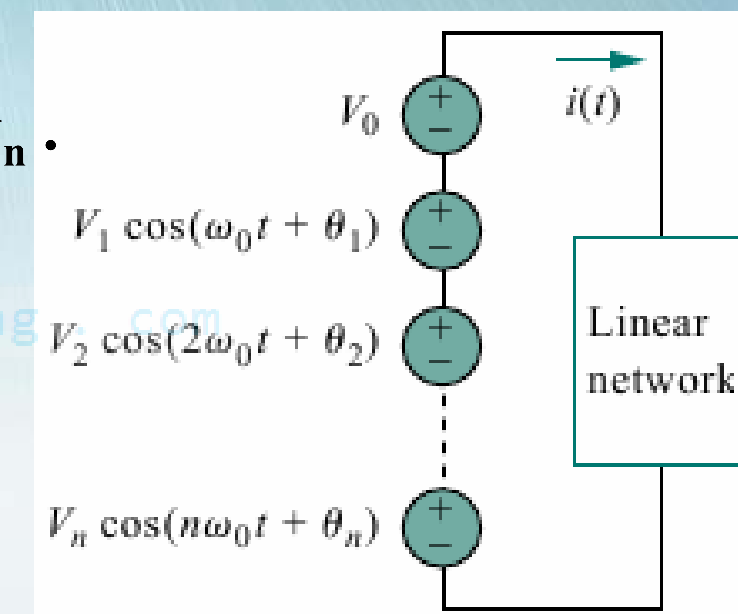
i. **Determine the Fourier Series of v(t):**



$$v(t) = V_0 + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t + \varphi_n)$$

ii. **Superposition :**

- **DC Analysis (ω=0): Find I_0.**
- **AC Analysis (ω=nω_0): Find I_n and ψ_n .**



$$i(t) = I_0 + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t + \psi_n)$$

❖ Using Frequency Response for Harmonic Analysis (AC analysis) :

- **Circuit Elements:** $R \quad L \quad C \Rightarrow R \quad jn\omega_0 L \quad \dfrac{1}{jn\omega_0 C}$

- **Determine Frequency Response: $H(jn\omega_0) = I(jn\omega_0)/V(jn\omega_0)$.**

- **From Fourier Series $\rightarrow V(jn\omega_0) \rightarrow I(jn\omega_0)$.**

$$H(jn\omega_0) = M(n\omega_0)\angle\theta(n\omega_0) \;\;\&\;\; V(jn\omega_0) = V_n\angle\varphi_n$$

$$\Rightarrow \boxed{I(jn\omega_0) = V_n.M(n\omega_0)\angle\varphi_n + \theta(n\omega_0) \;\; = I_n\angle\psi_n}$$

❖ Find Frequency Response $H(j\omega)$ * :

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega)$$

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}}$$

i. **Using the basic laws in the phasor domain .**

ii. **Assuming $Y(j\omega) = 1 \rightarrow$ Find $X(j\omega) \rightarrow$ then calculate $H(j\omega)$.**

iii. **Determine the Transfer Function H(s) (See chapter 6) $\rightarrow$ then replace <u>s</u> by <u>$j\omega$</u> .**

(*): very important to Electronic Engineering .