Chapter 2:

Fourier Transform

2.1: The Fourier Transform :

❖ Biến đổi Fourier cho tín hiệu không tuần hoàn f(t) : là một công cụ toán có phạm vi áp dụng rất lớn trong các bài toán kỹ thuật , nó được định nghĩa là một cặp biến đổi thuận – ngược như sau :

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}\,dt$$

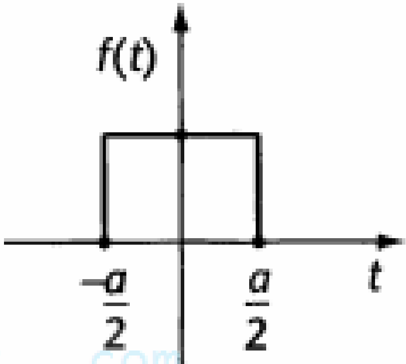
$$\mathrm{f(t)} = \mathcal{F}^{-1}\{\mathrm{F}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega).e^{j\omega t}\,d\omega$$

❖ Để có biến đổi Fourier, tín hiệu f(t) cũng phải thỏa mãn điều kiện Dirichlets.

❖ Example: Fourier Transform

Find the Fourier transform of

$$f(t) = \begin{cases} 0 & t < -a/2 \\ 1 & -a/2 < t < a/2 \\ 0 & a/2 < t \end{cases}$$



$$F(\omega) = \int_{-\infty}^{\infty} f(t).e^{-j\omega t}dt = \int_{-a/2}^{a/2} e^{-j\omega t}dt = \frac{e^{-\mathrm{j}\omega \mathrm{a}/2} - e^{\mathrm{j}\omega \mathrm{a}/2}}{-\mathrm{j}\omega}$$

$$F(\omega) = \frac{2\sin(\omega a/2)}{\omega} = \mathrm{a.Sa}\left(\frac{\omega a}{2}\right)$$

2.2: Properties of Fourier Transform :

1) Multiplication by a constant :

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then: $\boxed{\mathcal{F}\{k.f(t)\} = k.F(\omega)}$

2) Addition / Subtraction:

If: $\mathcal{F}\{f_1(t)\} = F_1(\omega)$ and $\mathcal{F}\{f_2(t)\} = F_2(\omega)$

Then: $\mathcal{F}\{f_1(t) \pm f_2(t)\} = F_1(\omega) \pm F_2(\omega)$

3) Translation in Time-domain:

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then: $$\mathcal{F}\{f(t - t_0)\} = F(\omega).e^{-j\omega t_0}$$

4) Translation in frequency-domain:

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then: $$\mathcal{F}\{f(t).e^{j\omega_0 t}\} = F(\omega - \omega_0)$$

5) Scale Changing:

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then: $$\mathcal{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{\omega}{a}\right)$$

$$\Rightarrow \quad \mathcal{F}\{f(-t)\} = F(-\omega)$$

6) Differentiation:

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then:

$$\mathcal{F}\left\{\frac{df(t)}{dt}\right\} = j\omega F(\omega)$$

$$\mathcal{F}\left\{f^{(n)}(t)\right\} = (j\omega)^n F(\omega)$$

Attention: **Equations are valid if $f(\pm\infty) = 0$**

7) Integration:

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then: $$\mathcal{F}\left\{\int_{-\infty}^{t} f(x)dx\right\} = \frac{F(\omega)}{j\omega} + \pi F(0)\delta(\omega)$$

Attention: The second term due to x(t) finite but $\int_{-\infty}^{\infty} f(x)dx \neq 0$

This term is zero if $\int_{-\infty}^{\infty} f(x)dx = 0$

8) Multiplication by t :

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then:

$$\mathcal{F}\left\{t.f(t)\right\} = j\frac{dF(\omega)}{d\omega}$$

$$\mathcal{F}\left\{t^n.f(t)\right\} = (j)^n\,\frac{d^nF(\omega)}{d\omega^n}$$

9) Modulation :

If: $\mathcal{F}\{f(t)\} = F(\omega)$

Then:

$$\mathcal{F}\left\{f(t).\cos(\omega_0 t)\right\} = \frac{1}{2}F(\omega-\omega_0)+\frac{1}{2}F(\omega+\omega_0)$$

10) Convolution in time-domain :

If: $Y(\omega) = X(\omega).H(\omega)$

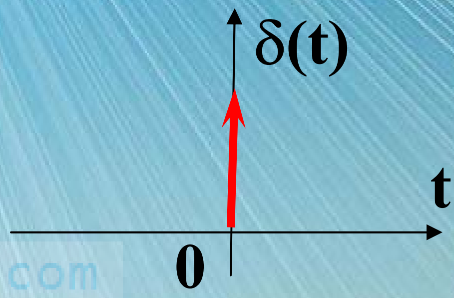
Then:

$$y(t) = \int_{-\infty}^{\infty} x(\tau).h(t-\tau)d\tau$$

2.3: Fourier Transform of Fundamental functions :

1) Impulse or Delta or Dirac Function :

$$\delta(t) = \begin{cases} \infty & \leftrightarrow \; for : t = 0 \\ 0 & \leftrightarrow \; for : t \neq 0 \end{cases}$$

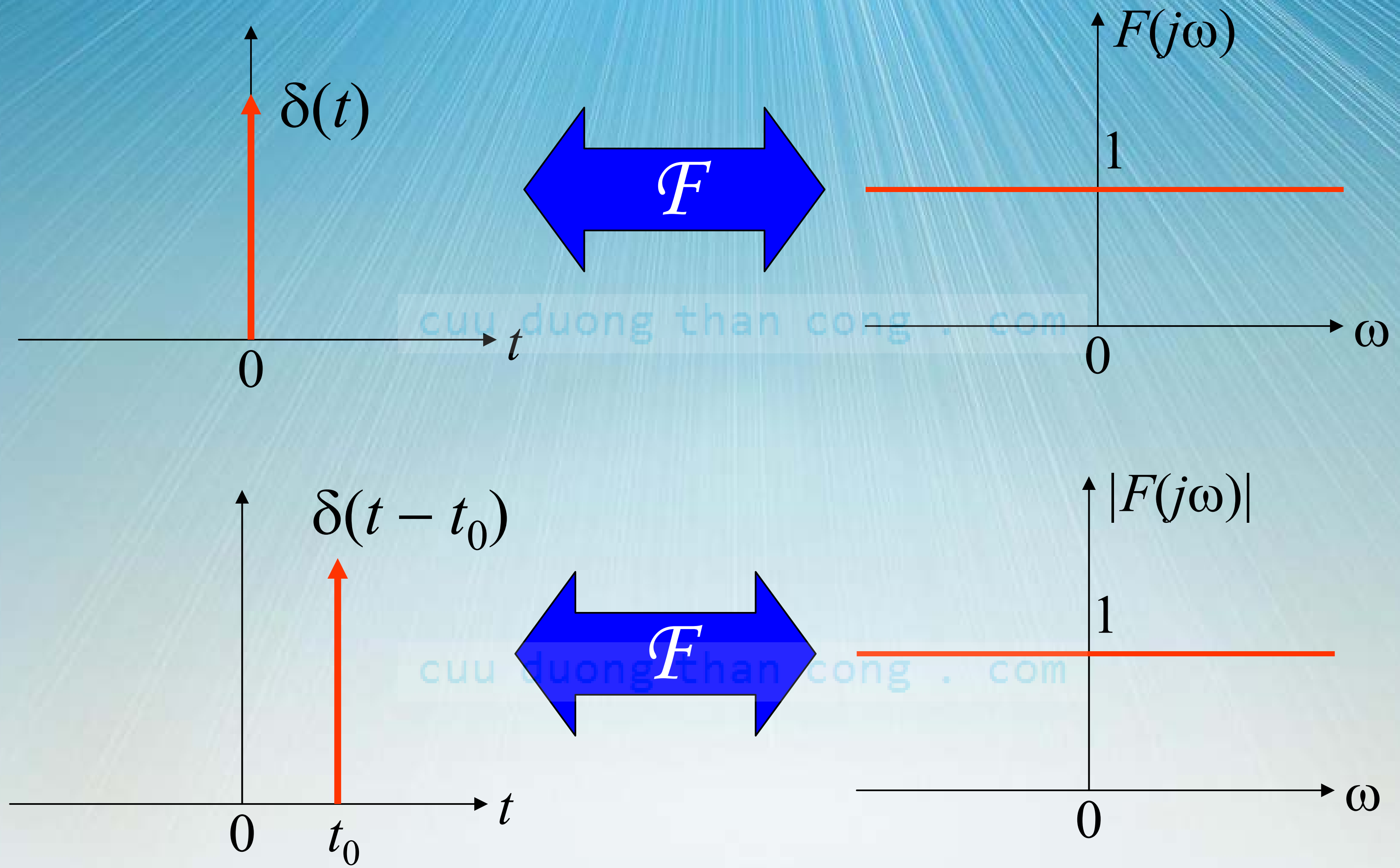


$$F(\omega) = \int_{-\infty}^{\infty} \delta(t).e^{-j\omega t} dt = e^{-j\omega 0} = 1$$

$$\boxed{\mathcal{F}\{\delta(t)\} = 1}$$

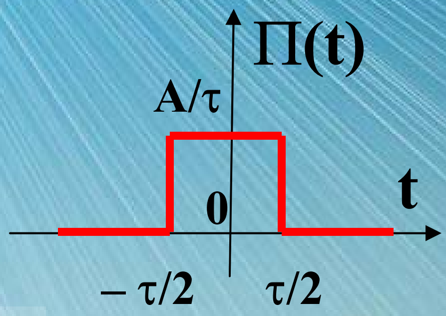
$$\boxed{\mathcal{F}\{\delta(t - t_0)\} = e^{-j\omega t_0}}$$

❖ Impulse Function :



2) Top-Hat Function :

$$\prod(t) = \begin{cases} 0 & \leftrightarrow for : t < -\tau/2 \\ A/\tau & \leftrightarrow for : -\tau/2 < t < \tau/2 \\ 0 & \leftrightarrow for : t > \tau/2 \end{cases}$$

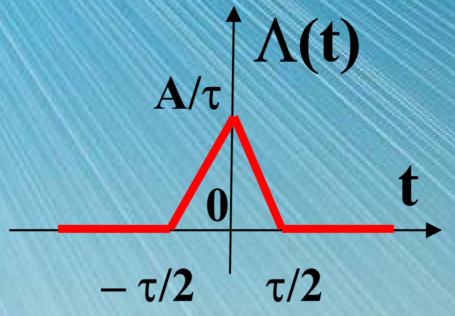


$$F(\omega) = \int_{-\tau/2}^{\tau/2} \frac{A}{\tau}.e^{-j\omega t} dt = A \frac{e^{-j\omega\tau/2} - e^{j\omega\tau/2}}{-j\omega\tau} = A \frac{2\sin(\omega\tau/2)}{\omega\tau}$$

$$\boxed{\mathcal{F}\left\{\prod(t)\right\} = A.Sa\left(\frac{\omega\tau}{2}\right)}$$

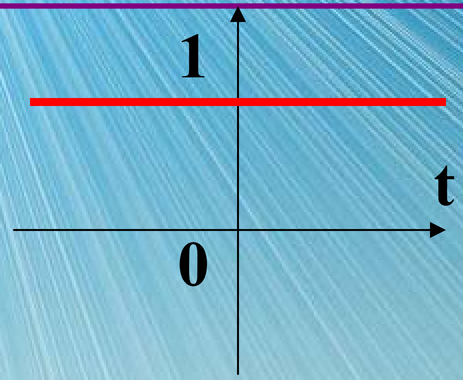
3) Triangle Function :

$$\Lambda(t) = \begin{cases} \frac{2A}{\tau^2}\left(t+\tau/2\right) & \leftrightarrow for: -\tau/2 < t < 0 \\ \frac{2A}{\tau^2}\left(-t+\tau/2\right) & \leftrightarrow for: 0 < t < \tau/2 \\ 0 & \leftrightarrow for: |t| > \tau/2 \end{cases}$$



4) Constant Function :

$$\boxed{\mathcal{F}\{1\} = 2\pi.\delta(\omega)}$$



Proof:

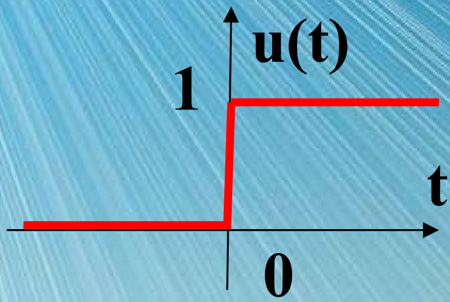
$$\mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi.\delta(\omega).e^{j\omega t}\,d\omega = e^{j\omega t}\Big|_{\omega=0} = 1$$

$$\Rightarrow \boxed{\mathcal{F}\left\{e^{j\omega_0 t}\right\} = 2\pi.\delta(\omega - \omega_0)}$$

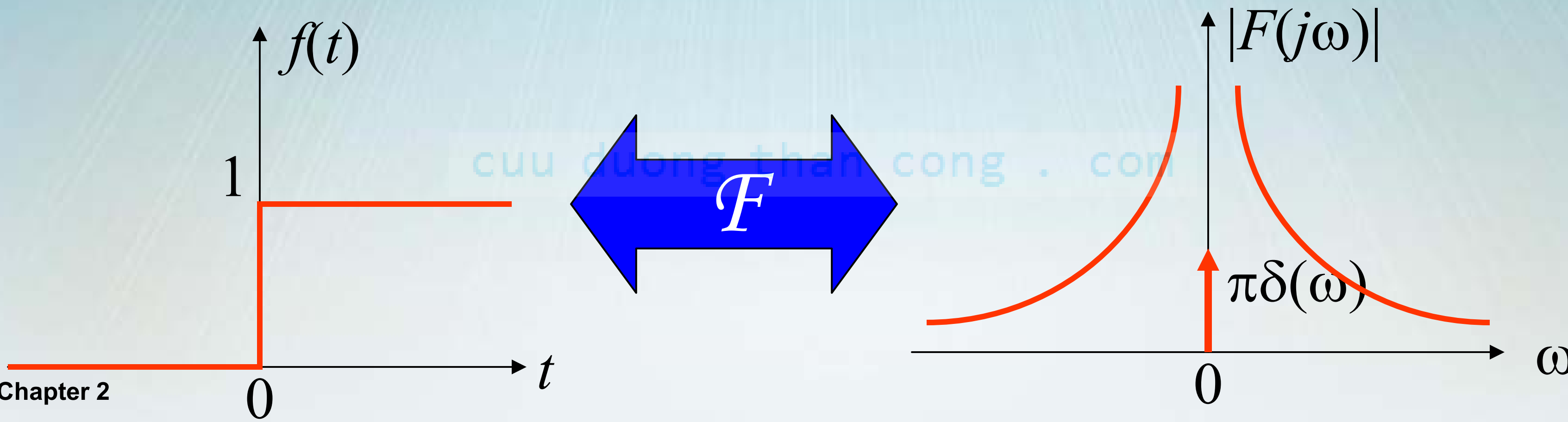
$$\Rightarrow \boxed{\mathcal{F}\left\{e^{-j\omega_0 t}\right\} = 2\pi.\delta(\omega + \omega_0)}$$

5) Unit Step Function u(t):

$$u(t) = \begin{cases} 1 \leftrightarrow for: t > 0 \\ 0 \leftrightarrow for: t < 0 \end{cases}$$

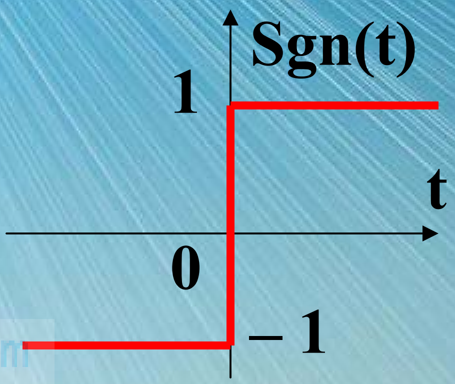


$$\boxed{\mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)}$$



6) Signum Function sgn(t) :

$$\operatorname{sgn}(t) = \begin{cases} 1 \leftrightarrow & for : t > 0 \\ -1 \leftrightarrow & for : t < 0 \end{cases}$$



$$\operatorname{sgn}(t) = 2u(t) - 1$$

$$\mathcal{F}\{\operatorname{sgn}(t)\} = \frac{2}{j\omega} + 2\pi\delta(\omega) - 2\pi\delta(\omega)$$

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$$\boxed{\mathcal{F}\{\operatorname{sgn}(t)\} = \frac{2}{j\omega}}$$

❖ Fourier Transform Pairs :

f(t)	F(ω)
u(t)	$\frac{1}{j\omega} + \pi\delta(\omega)$
δ(t)	1
1 (DC source)	2πδ(ω)
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
e^{-at}.u(t) & e^{at}.u(– t)	$\frac{1}{a+j\omega}$ & $\frac{1}{a-j\omega}$
Sgn(t)	$\frac{2}{j\omega}$

❖ Fourier Transform Pairs :

f(t)	F(ω)
AC source: $\cos(\omega_0 t)$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
AC source: $\sin(\omega_0 t)$	$-j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
AC transient : $\cos(\omega_0 t).u(t)$	$\frac{\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]+\frac{j\omega}{\omega_0^2-\omega^2}$
AC transient : $\sin(\omega_0 t).u(t)$	$-j\frac{\pi}{2}\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]+\frac{\omega_0}{\omega_0^2-\omega^2}$
two-side: $e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+\omega^2}=\frac{1}{-j\omega+\alpha}+\frac{1}{j\omega+\alpha}$

2.4: Using MATLAB to Compute Fourier Transform :

```
syms  t w ;
ft = exp(-t^2/2) ;
Fw = fourier(ft);
pretty(Fw);
```

2.5: Applications of Fourier Transform :

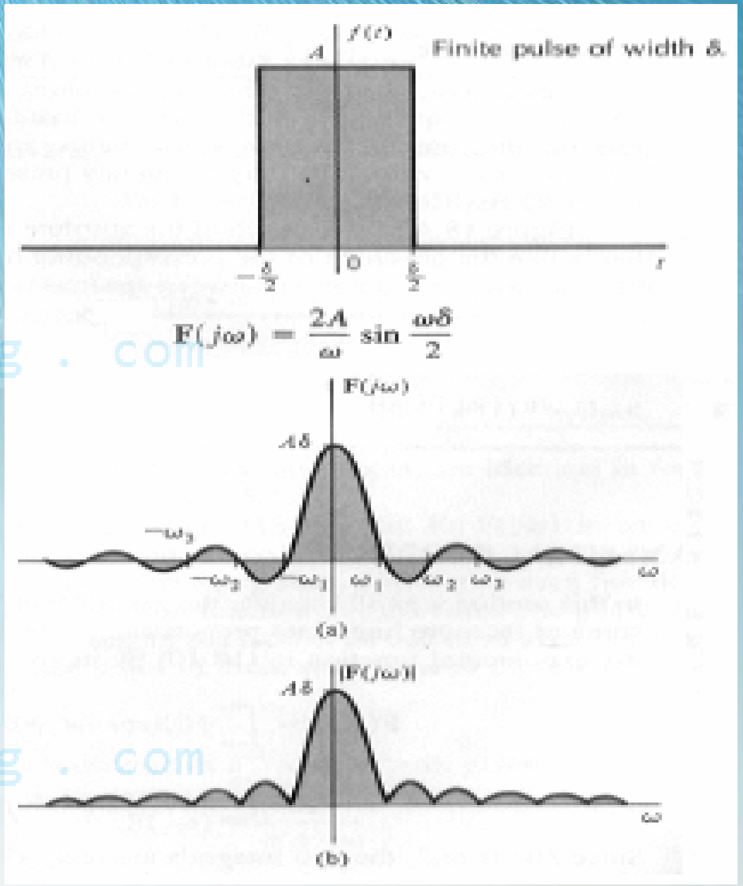
1) Frequency Spectra:

❖ **Từ hàm :** $F(\omega) = \left|F(\omega)\right| e^{j\varphi(\omega)}$

➢ **Phổ biên độ: biểu diễn |F(jω)| theo ω .**

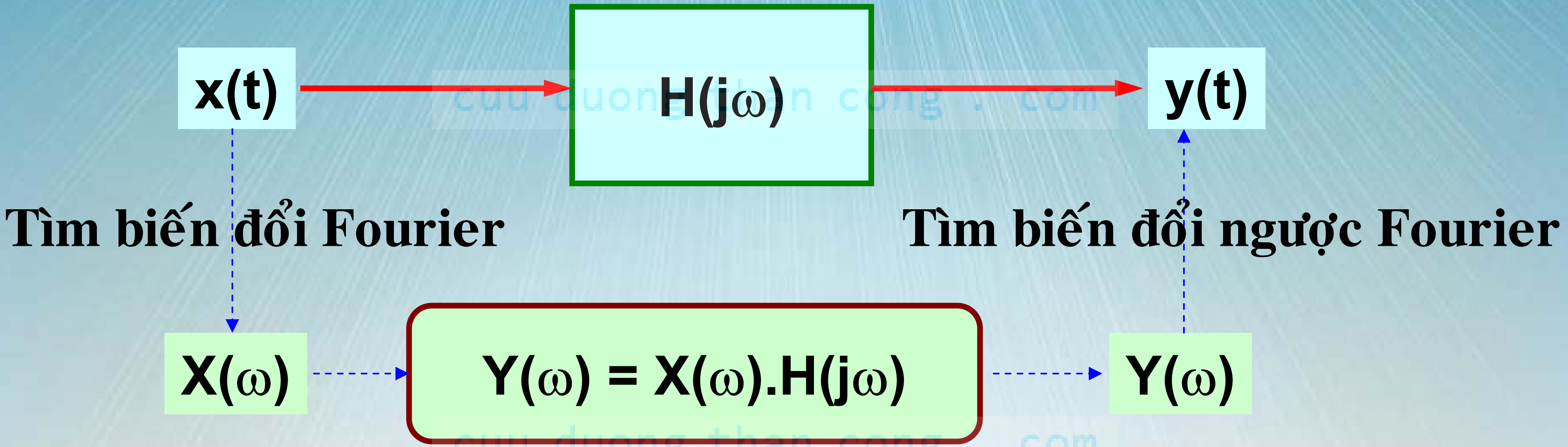
➢ **Phổ pha : φ(ω) theo ω .**

❖ **Phổ biên độ và phổ pha của tín hiệu không tuần hoàn là các hàm liên tục theo ω .**



2) Circuit Analysis :

❖ Biến đổi Fourier dùng cho các bài toán mạch mà có thông số phụ thuộc tần số.



❖ Circuit elements :

$$R \longrightarrow R$$

$$L \longrightarrow j\omega L$$

$$C \longrightarrow \frac{1}{j\omega C}$$

❖ Determine transfer function $H(j\omega)$:

Using the propotional theorem to apply the ladder network

i. **Transfer the circuit to frequency domain.**

ii. **Assume $Y(\omega) = 1$.**

iii. **Determine $X(\omega)$ using the circuit laws.**

iv. **Transfer Function $H(j\omega) = Y(\omega)/X(\omega)$.**