Chapter 3:

Laplace Transform

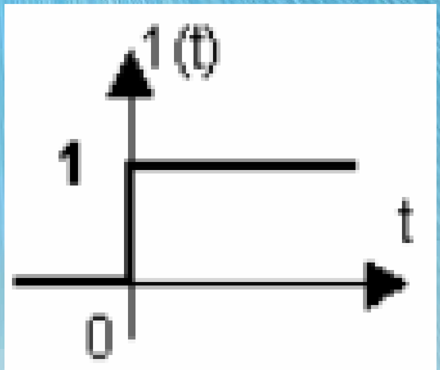
3.1: Definition :

$$\mathfrak{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt$$

- **f(t) : original function.**
- **F(s) : Laplace transform.**

❖Example1: Find Laplace Transform ?

Unit Step Function u(t) or 1(t) :

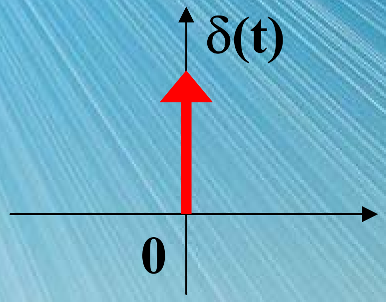


$$u(t) = \begin{cases} 1 \leftrightarrow \ for: t > 0 \\ 0 \leftrightarrow \ for: t < 0 \end{cases}$$

$$\mathfrak{L}\{u(t)\} = \int_0^\infty e^{-st} dt = \left. \frac{e^{-st}}{-s} \right|_0^\infty = \frac{1}{s}$$

❖Example2: Find Laplace Transform ?

Impulse or Delta or Dirac Function δ(t) :



$$\delta(t)=\begin{cases}\infty \leftrightarrow \ for: t=0 \\ 0 \leftrightarrow \ for: t\neq 0\end{cases}$$

$$\mathfrak{L}\{\delta(\mathrm{t})\}=\int_0^\infty \delta(t)e^{-st}dt=e^0=1$$

3.2: Properties of Laplace Transform :

1) Multiplication by a constant :

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then: $\boxed{\mathfrak{L}\{k.f(t)\} = k.F(s)}$

2) Addition / Subtraction:

If: $\mathfrak{L}\{f_1(t)\} = F_1(s)$ and $\mathfrak{L}\{f_2(t)\} = F_2(s)$

Then: $\mathfrak{L}\{f_1(t) \pm f_2(t)\} = F_1(s) \pm F_2(s)$

3) Translation in Time-domain:

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then: $\mathfrak{L}\{f(t - t_0).u(t - t_0)\} = F(s).e^{-st_0}$

4) Translation in frequency-domain:

If: $$\mathfrak{L}\{f(t)\} = F(s)$$

Then: $$\mathfrak{L}\{f(t).e^{-at}\} = F(s+a)$$

5) Scale Changing:

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then: $$\mathfrak{L}\{f(at)\} = \frac{1}{a} F\left(\frac{s}{a}\right)$$

6) Differentiation:

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then:
$$\mathfrak{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$

$$\mathfrak{L}\left\{\frac{d^2f(t)}{dt^2}\right\} = s^2F(s) - s.f(0) - f'(0)$$

$$\mathfrak{L}\left\{f^{(n)}(t)\right\} = s^nF(s) - s^{n-1}.f(0) - \ldots - f^{(n-1)}(0)$$

7) Integration:

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then: $$\mathfrak{L}\left\{\int_0^t f(x)dx\right\} = \frac{F(s)}{s}$$

8) Multiplication by t :

If: $\mathcal{L}\{f(t)\} = F(s)$

Then: $$\mathcal{L}\left\{t.f(t)\right\} = -\frac{dF(s)}{ds}$$

$$\mathcal{L}\left\{t^n.f(t)\right\} = (-1)^n \frac{d^nF(s)}{ds^n}$$

9) Division by t :

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then:

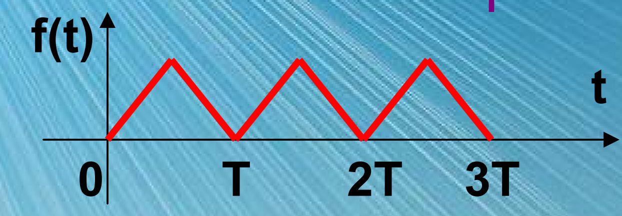
$$\mathfrak{L}\left\{\frac{f(t)}{t}\right\} = \int_{s}^{\infty} F(x)\,dx$$

10) Periodic signal :

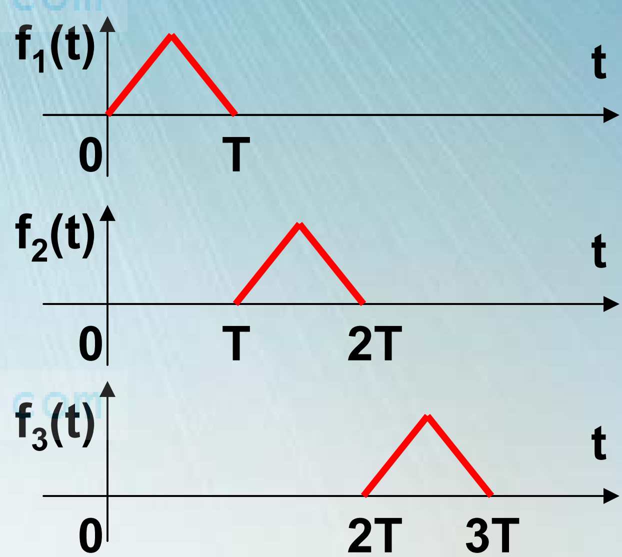
$$f(t) = f_1(t) + f_2(t) + f_3(t) + \ldots$$

$$= f_1(t) + f_1(t-T)u(t-T) + f_1(t-2T)u(t-2T) + \ldots$$

$$F(s) = F_1(s) + F_1(s).e^{-sT} + F_1(s).e^{-s2T} + \ldots$$



A periodic function



Decomposition of f(t)

Then: $$\boxed{F(s) = \frac{F_1(s)}{1-e^{-sT}}}$$

$F_1(s)$ = Laplace Transform over first period only.

11) Initial Value Theorem :

If: $\mathfrak{L}\{f(t)\} = F(s)$

Then: $$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty}\big(s.F(s)\big)$$

12) Final Value Theorem :

If: $\mathfrak{L}\{f(t)\} = F(s)$

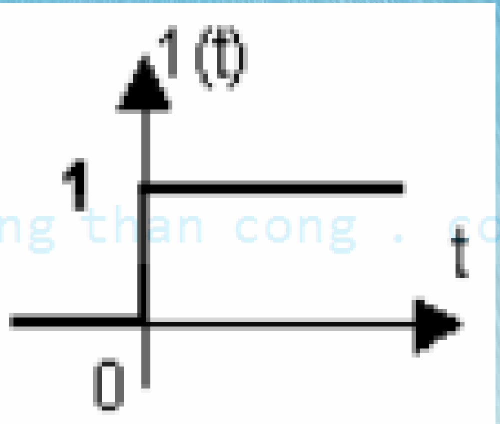
Then: $$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0}\left(s.F(s)\right)$$

Attention: **all poles of F(s) must be located in the left half of the s-plane, except s = 0 .**

3.3: Laplace Transform of Fundamental functions :

1) Unit Step Function u(t) or 1(t) :

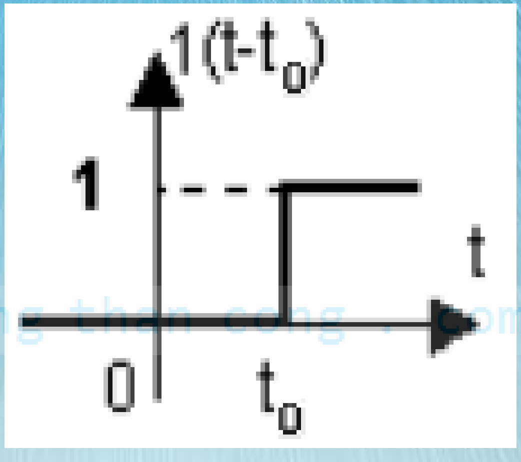
$$u(t)=\begin{cases}1 \leftrightarrow \; for: t>0 \\ 0 \leftrightarrow \; for: t<0\end{cases}$$



$$\boxed{\pounds\{u(t)\} = \frac{1}{s}}$$

2) Translated unit step $u(t - t_0)$:

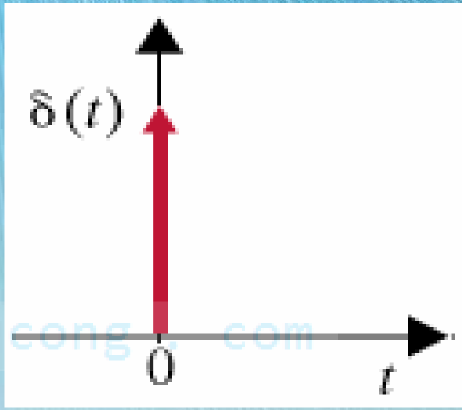
$$u(t-t_0)=\begin{cases}1 \leftrightarrow for: t > t_0 \\ 0 \leftrightarrow for: t < t_0\end{cases}$$



$$\pounds\{u(t-t_0)\} = \frac{1}{s}e^{-st_0}$$

3) Impulse or Delta Function $\delta(t)$:

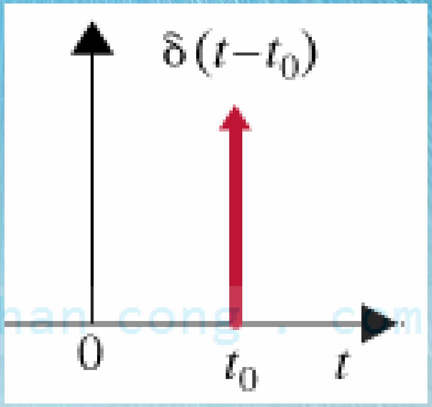
$$\delta(t) = u'(t) = \begin{cases} 0 \leftrightarrow for : t \neq 0 \\ \infty \leftrightarrow for : t = 0 \end{cases}$$



$$\boxed{£\{\delta(t)\} = 1}$$

4) Translated Delta Function $\delta(t - t_0)$:

$$\delta(t-t_0)=\begin{cases}0 \leftrightarrow for: t \neq t_0 \\ \infty \leftrightarrow for: t = t_0\end{cases}$$



$$\pounds\{\delta(t-t_0)\} = e^{-st_0}$$

5) Decaying Exponential Function e^{-at} :

$$\mathfrak{L}\{e^{-at}\} = \frac{1}{s+a}$$

6) Sinusoidal Function :

$$\mathfrak{L}\{\cos(\omega t)\} = \frac{s}{s^2 + \omega^2}$$

$$\mathfrak{L}\{\sin(\omega t)\} = \frac{\omega}{s^2 + \omega^2}$$

7) Power Function :

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

3.4: Laplace Transform Pairs :

$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

f(t)	F(s)
te^{-at}	$\dfrac{1}{(s+a)^2}$
$\sin(\omega t)$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\dfrac{s}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\dfrac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)$	$\dfrac{\omega}{(s+a)^2+\omega^2}$

3.5: Methods of Finding Laplace Trans:

1) **Direct Method :**

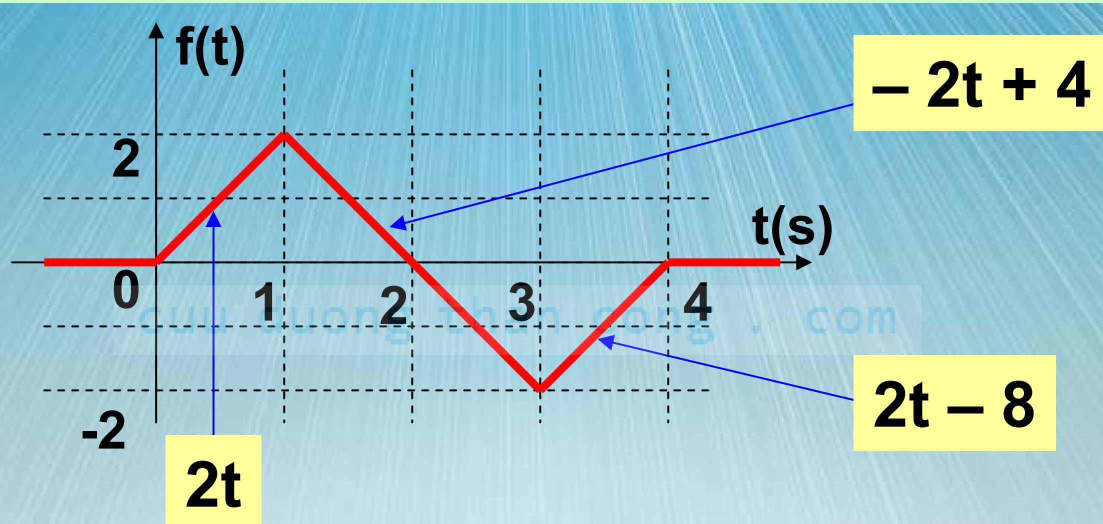
2) **Use of Tables :**

3) **Series Method :** $\text{Let } f(t) = a_0 + a_1 t + a_2 t^2 + \ldots$

Then: $\mathcal{L}\{f(t)\} = \dfrac{a_0}{s} + \dfrac{1!a_1}{s^2} + \dfrac{2!a_2}{s^3} + \ldots$

4) Use unit step functions to represent a Finite Function :

Use step functions to write an expression for the function:



The expression for f(t) is :

$$f(t) = 2t[u(t) - u(t-1)] + (-2t+4)[u(t-1) - u(t-3)] + (2t-8)[u(t-3) - u(t-4)]$$

- *Need to change for using the table of Laplace Transform Pairs.*

3.6: MATLAB to find Laplace Trans:

```
syms  s t ;
Fs = laplace(sin(4t));
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