Chapter 4:

Inverse Laplace Transform

4.1: Definition :

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

- **f(t) : original function , is unique .**
- **F(s) : Laplace transform.**

4.2: Inverse Laplace Transform of Fundamental functions :

Corresponding on Chapter 3

4.3: Properties of Inverse Laplace Transform :

1) **Linearity :**

$$\mathfrak{L}^{-1}\{C_1F_1(s)+C_2F_2(s)\}=C_1f_1(t)+C_2f_2(t)$$

2) **First Translation :**

$$\mathfrak{L}^{-1}\{F(s-a)\}=e^{at}f(t)$$

3) **Second Translation :**

$$\mathfrak{L}^{-1}\{F(s).e^{-sT}\}=f(t-T).u(t-T)$$

4.3: Properties of Inverse Laplace Transform :

4) Scale Changing :

$$\mathcal{L}^{-1}\{F(ks)\} = \frac{1}{k} f\left(\frac{t}{k}\right)$$

4.3: Properties of Inverse Laplace Transform :

5) Inverse Laplace Transform of devivatives :

$$\mathfrak{L}^{-1}\{F^{(n)}(s)\} = (-1)^n t^n f(t)$$

6) Inverse Laplace Transform of Integrals :

$$\mathfrak{L}^{-1}\{\int_s^\infty F(u)du\} = \frac{f(t)}{t}$$

7) Multiplication by s:

$$\mathcal{L}^{-1}\{s.F(s)\} = f'(t)$$

Attention:

$$\mathcal{L}^{-1}\{s.F(s) - f(0)\} = f'(t) - f(0).\delta(t)$$

8) Division by s :

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)du$$

9) The convolution:

$$\mathfrak{L}^{-1}\{F(s), G(s)\} = f(t) * g(t) = \int_0^t f(x).g(t-x)dx$$

- **f(t)*g(t) : the convolution of f(t) and g(t).**

4.4: Methods of Finding Inverse Laplace Transform:

1) Use of Tables :

2) Series Method : $$\text{Let } F(s) = \frac{a_0}{s} + \frac{a_1}{s^2} + \frac{a_2}{s^3} + \ldots$$

Then: $$f(t) = a_0 + \frac{a_1 t}{1!} + \frac{a_2 t^2}{2!} + \ldots$$

3) Partial Fraction Method :

- Rational Function P(s)/Q(s) can be written as the sum of rational functions having form:

$$\frac{A}{(as+b)^r} \text{ or } \frac{(As+B)}{(as^2+bs+c)^r}$$

- The constants A, B, C … can be found by equating of like powers of s on both side / or substitution the values of s .

4) The Heavyside Expansion Formula :

Let F(s) = P(s)/Q(s)

a) Q(s) has n simple zeros: $s_1, s_2, \ldots, s_n$.

$$\boxed{f(t) = \sum_{i=1}^{n} K_i e^{s_i t}}$$

$$K_i = \lim_{s \to s_i} \left(\frac{P(s)}{Q(s)} (s - s_i) \right) = \left. \frac{P(s)}{Q'(s)} \right|_{s = s_i}$$

b) Q(s) has s_1 multiple zero rth-order:

$$\frac{P(s)}{Q(s)} = \frac{K_{1,1}}{(s-s_1)} + \frac{K_{1,2}}{(s-s_1)^2} + \ldots + \frac{K_{1,r}}{(s-s_1)^r} + \frac{K_{r+1}}{s-s_{r+1}} + \ldots + \frac{K_n}{s-s_n}$$

$$K_{1,k} = \frac{1}{(r-k)!}\frac{d^{r-k}}{ds^{r-k}}\left[\frac{P(s)}{Q(s)}(s-s_1)^r\right]\Bigg|_{s=s_1;k=1\div r}$$

For finding the original function we use:

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s-s_1)^r}\right\} = \frac{1}{(r-1)!}t^{r-1}e^{s_1 t}$$

c) Q(s) has complex zeros:

$$\mathbf{s_{1,2} = \alpha \pm i\beta}$$

$$f(t) = 2\,\mathrm{Re}\left\{\frac{P(s_1)}{Q'(s_1)}e^{s_1 t}\right\} + \sum_{i=3}^{n} K_i e^{s_i t}$$

4.5: MATLAB to find Inverse Laplace Transform:

```
syms  s t ;
Fs = (3*s+2)/(s^2+3*s+2);
ft = ilaplace(Fs);
pretty(ft);
```