Chapter 5:

Applications The Laplace Transform to Differential Equations

5.1: ODE with constant coefficients :

- **We wish to solve the equation:**

$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_0 y = f(t) \qquad (1)$$

to the initial conditions (IC) :

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \\ \ldots \\ y^{(n-1)}(0) = y_{n-1} \end{cases}$$

General Procedure:

i. Taking the Laplace Transform of both sides. → Equation in s-space

ii. Determine Y(s) .

iii. Using Inverse Laplace Transform of finding the original function y(t) .

Recall:

$$\mathfrak{L}\left\{y^{(n)}(t)\right\} = s^{n}Y\left(s\right) - s^{n-1}.y(0) - \ldots - y^{(n-1)}(0)$$

❖Example1: Solve ODE by Laplace Transform

$$\text{Solve} \quad y'' - y = t \quad \text{with} \quad y(0) = 1;\, y'(0) = 1$$

Step1: **Laplace Transform** $s^2 Y - sy(0) - y'(0) - Y = 1/s^2$

Step2: **Solving in s-space and partial fraction expansion gives:**

$$Y(s) = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

Step3: **we obtain the solution**

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = e^t + \sinh t - t$$

5.2: System of ODEs :

- We wish to solve the state variable equation:

$$\begin{cases} \dfrac{dx}{dt} = Ax + By \\ \dfrac{dy}{dt} = Cx + Dy \end{cases}$$

to the initial conditions (IC) :

$$\begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

- Applying : $\boxed{\mathfrak{L}\{dx/dt\} = sX(s) - x_0}$ to solve X(s) & Y(s).

- Find x(t) and y(t) using Inverse Laplace Transform.

❖Example: System of ODEs

Solve $\begin{cases} y_1' = -\frac{8}{100}y_1 + \frac{2}{100}y_2 + 6 \\ y_2' = \frac{8}{100}y_1 + \frac{8}{100}y_2 \end{cases}$ with $\begin{cases} y_1(0) = 0 \\ y_2(0) = 150 \end{cases}$

$$\begin{cases} (s+0.08)\mathrm{Y}_1 - 0.02\mathrm{Y}_2 = \frac{6}{s} \\ -0.08\mathrm{Y}_1 + (s-0.08)\mathrm{Y}_2 = 150 \end{cases}$$

$$\mathrm{Y}_1 = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$

$$\mathrm{Y}_2 = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$
$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}.$$

5.3: Applications to Mechanics :