

Chapter 5:

Applications of The Laplace Transform to Differential Equations



5.1: ODE with constant coefficients :

- We wish to solve the equation:

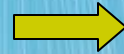
$$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = f(t) \quad (1)$$

to the initial conditions (IC) :

$$\begin{cases} y(0) = y_0 \\ y'(0) = y_1 \\ \dots \\ y^{(n-1)}(0) = y_{n-1} \end{cases}$$

■ General Procedure:

i. Taking the Laplace Transform of both sides.



Equation in s-space

ii. Determine $Y(s)$.

iii. Using Inverse Laplace Transform of finding the original function $y(t)$.

Recall:

$$\mathcal{L}\{y^{(n)}(t)\} = s^n Y(s) - s^{n-1} \cdot y(0) - \dots - y^{(n-1)}(0)$$

❖ Example 1: Solve ODE by Laplace Transform

Solve $y'' - y = t$ with $y(0) = 1; y'(0) = 1$

Step 1: Laplace Transform $s^2 Y - sy(0) - y'(0) - Y = 1/s^2$

Step 2: Solving in s-space and partial fraction expansion gives:

$$Y(s) = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$$

Step 3: we obtain the solution

$$y(t) = \mathcal{L}^{-1} \{Y(s)\} = e^t + \sinh t - t$$

5.2: System of ODEs :

- We wish to solve the state variable equation:

$$\begin{cases} \frac{dx}{dt} = Ax + By \\ \frac{dy}{dt} = Cx + Dy \end{cases}$$

to the initial conditions (IC) :

$$\begin{cases} x(0) = x_0 \\ y(0) = y_0 \end{cases}$$

- Applying : $\mathcal{L}\{dx/dt\} = sX(s) - x_0$ to solve $X(s)$ & $Y(s)$.
- Find $x(t)$ and $y(t)$ using Inverse Laplace Transform.

❖ Example: System of ODEs

$$\text{Solve } \begin{cases} y_1' = -\frac{8}{100} y_1 + \frac{2}{100} y_2 + 6 \\ y_2' = \frac{8}{100} y_1 + \frac{8}{100} y_2 \end{cases}$$

$$\text{with } \begin{cases} y_1(0) = 0 \\ y_2(0) = 150 \end{cases}$$

$$\begin{cases} (s + 0.08)Y_1 - 0.02Y_2 = \frac{6}{s} \\ -0.08Y_1 + (s - 0.08)Y_2 = 150 \end{cases}$$

$$Y_1 = \frac{100}{s} - \frac{62.5}{s+0.12} - \frac{37.5}{s+0.04}$$

$$Y_2 = \frac{100}{s} + \frac{125}{s+0.12} - \frac{75}{s+0.04}$$

$$y_1 = 100 - 62.5e^{-0.12t} - 37.5e^{-0.04t}$$

$$y_2 = 100 + 125e^{-0.12t} - 75e^{-0.04t}$$

5.3: Applications to Mechanics :

