Chapter 7:

Analytic Function

7.1: Complex number

a) The real number:

- Natural number = positive integers.
- negative integers and zero .
- Rational number: $\frac{1}{3}, \frac{2}{3}, \frac{3}{8} \ldots$.
- Irrational number: $\pi = 3.14159\ldots$.
- Rational and irrational number = real number .
- Real number = represented by real axis .

b) The Complex Number:

- **Imaginary number = root of negative real number .**

$$\sqrt{-3} = \sqrt{3}\sqrt{-1} = \sqrt{3}\,j$$

- **Complex number = real number + Imaginary number .**

$$z = a + jb$$

- **The components:**

$$\mathrm{Re}(z) = a$$

$$\mathrm{abs}(z) = \sqrt{a^2 + b^2}$$

$$\mathrm{Im}(z) = b$$

$$\arg(z) = \tan^{-1}(b/a)$$

$$(-\pi < \arg(z) \le \pi)$$

c) The forms of Complex Number:

- **Algebra form:** $z = a + jb$
- **Exponential form:** $z = r.e^{j\theta}$
- **Trigonometric form:** $z = r[\cos\theta + j\sin\theta]$
- **Steinmetz form:** $z = r\angle\theta$

d) The Complex Conjugate and rules:

- **Definition:** $\bar{z} \text{ or } \overset{*}{z} = a - jb = re^{-j\theta}$

- **Rules:**

$$z.(\overset{*}{z}) = |z|^2$$

$$(\overset{*}{z.w}) = \overset{*}{z}.\overset{*}{w}$$

$$(\overset{*}{\frac{z}{w}}) = \frac{\overset{*}{z}}{\overset{*}{w}}$$

$$(\overset{*}{z + w}) = \overset{*}{z} + \overset{*}{w}$$

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e) The Fundamental Operations:

1. Addition:

2. Subtraction:

3. Multiplication :

4. Division :

5. Powers:

6. The nth Roots:

7.2: Function of a complex variable

❖ **Function = one of the most important concepts in math.**

❖ **Function = rule hat assigns one element in set A to one/only one element in set B.**

❖ **Complex Function (or *Function of a complex variable*) $w = f(z)$: input z and output w are complex.**

▪ **Example:** $w = f(z) = z^2 - (2+j)z$ **defines a complex function.**

❖ **Domain S of a complex function = set of all complex numbers z for which f(z) defined.**

- **Types :** → **Single-valued: $f(z) = z^2$. (default)**
  → **Multi-valued: f(z) = sqrt(z) .**

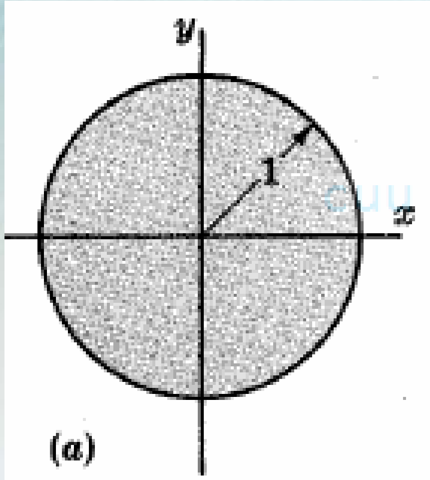
❖ Example1: Function complex variable

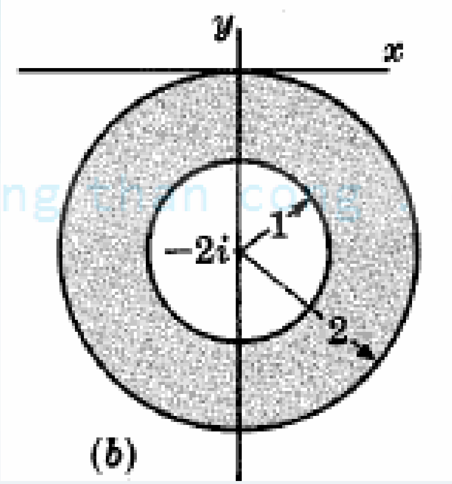
Determine region represented by: (a) $|z| < 1$ (b) $1 < |z + j2| \le 2$ (c) $\pi/3 \le \arg(z) \le \pi/2$?

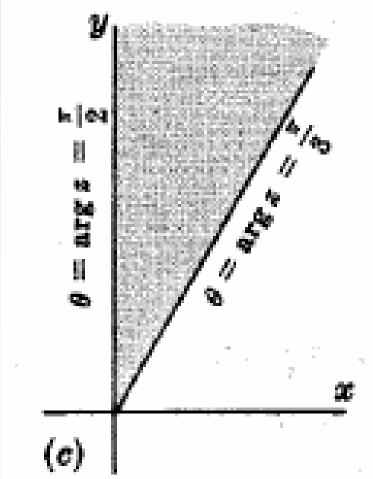
a) $|z| < 1$: Interior of a circle radius 1.

b) $1 < |z+j2| \le 2$: exterior radius 1 but interior of a circle radius 2.

c) $\pi/2 \le \arg(z) \le \pi/2$: infinite region, including the lines.





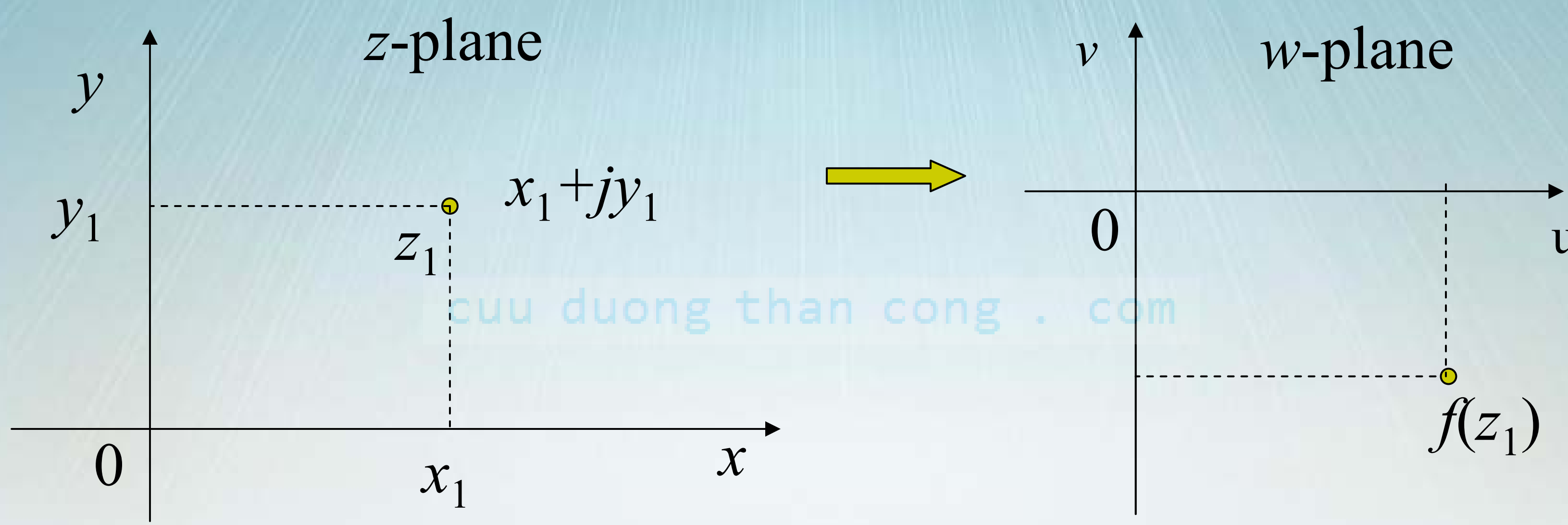


❖ Real and Imaginary Parts :

Every z on S , we have w = $f(z)$ = u +jv = u(x,y) + jv(x,y)

Real part — Imaginary part

❖ **Complex Function = mapping or transformation. Each point (region) in z-plane is mapped onto another in w-plane.**



❖Example2: Function complex variable

What are the real and imaginary parts of $f(z) = z + \frac{1}{z}$?

- **Let z = x + jy, we have:**

$$f(z) = x + jy + \frac{1}{x + jy} = x + jy + \frac{x - jy}{x^2 + y^2}$$

$$u(x,y) = x + \frac{x}{x^2 + y^2}$$

$$v(x,y) = y - \frac{y}{x^2 + y^2}$$

❖Example3: In polar representation

What are the real and imaginary parts of $f(z) = z + \frac{1}{z}$ in polar representation ?

- **Let z = re$^{j\theta}$, we have:**

$$f(z) = re^{j\theta} + \frac{1}{re^{j\theta}} = r(\cos\theta + j\sin\theta) + \frac{1}{r}(\cos\theta - j\sin\theta)$$

$$\Rightarrow \quad \boxed{u(r,\theta) = r\cos\theta + \frac{\cos\theta}{r}}$$

$$\boxed{v(r,\theta) = r\sin\theta - \frac{\sin\theta}{r}}$$

7.3: Limit and Continuity

1) Limit:

- $f(z)$ said to have the limit L at z_0 if :

for all $z \neq z_0$ in the disk $|z - z_0| < \delta$ we have: $$|f(z) - L| < \varepsilon$$

(δ, ε : positive real number)

- Limit is written as : $$\lim_{z \to z_0} f(z) = L$$

- if Limit exists, it is unique.

Properties:

Limit of complex function has the same properties in the real case.

$$\lim_{z\to a}\left(f(z)\pm g(z)\right)=\lim_{z\to a}f(z)\pm\lim_{z\to a}g(z)$$

$$\lim_{z\to a}\left(f(z).g(z)\right)=\lim_{z\to a}f(z).\lim_{z\to a}g(z)$$

$$\lim_{z\to a}\left(\frac{f(z)}{g(z)}\right)=\frac{\lim\limits_{z\to a}f(z)}{\lim\limits_{z\to a}g(z)}$$

❖Example1: Limit

Evaluate $\lim\limits_{z \to j2} (z^2 + 2z + 5)$?

$$\lim_{z \to j2}\left(z^2 + 2z + 5\right) = -4 + j4 + 5$$

$$\boxed{\lim_{z \to j2}\left(z^2 + 2z + 5\right) = 1 + j4}$$

2) Continuity:

- $f(z)$ is continuous at z_0 if :
  - i. $f(z_0)$ must exist.
  - ii. $\lim\limits_{z \to z_0} f(z) = L$ must exist.
  - iii. $L = f(z_0)$.

And written by : $\lim\limits_{z \to z_0} f(z) = f(z_0)$

❖Example1: Continuity

Show that $f(z) = z^3$ is continuous at $z = j$?

i) $f(z = j) = j^3 = -j$: exists

ii) $\lim\limits_{z \to j} z^3 = -j$

$\Rightarrow$ $f(z) = z^3$ is continuous at $z = j$

7.4: Derivative:

- The derivative of a complex function f(z), written f'(z), defined :

$$f'(z) = \lim_{\Delta z \to 0}\left(\frac{f(z+\Delta z) - f(z)}{\Delta z}\right)$$

❖ Example: Given $f(z) = z^2$, find f'(z) ?

$$f(z+\Delta z) - f(z) = z^2 + 2z\Delta z + \Delta z^2 - z^2$$

$$\lim_{\Delta z \to 0}\frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0}(2z + \Delta z) = 2z$$

❖ Properties:

$$(c.f)' = c.f'$$

$$(f+g)' = f' + g'$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - f.g'}{g^2}$$

$$(f.g)' = f'g + f.g'$$

If $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$ then: $\lim\limits_{z \to z_0} \frac{f(z)}{g(z)} = \lim\limits_{z \to z_0} \frac{f'(z)}{g'(z)}$

(L'Hopital's rule)

7.5: Cauchy – Riemann equations :

- **Let f(z) = u(x,y) + jv(x,y) differentiable at z = x + jy , u and v exist and satify the Cauchy – Riemann equations :**

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases}$$

7.6: Analytic function and properties:

a) Analytic Function:

- if f(z) = defined and differentiable in domain S : f(z) is analytic in S .

❖ **Example:** Is the function $f(z) = z^2$ analytic ?

We have: $f(z) = x^2 - y^2 + j2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial x} = 2y$$

⇒ **f(z) = analytic**

b) The Test :

- Necessary and sufficient conditions to be analytic are :
  - f (z) must be defined.
  - first-order partial derivatives (u_x, u_y, v_x, v_y) must exist.
  - Cauchy-Riemann equations must be satisfied.

c) Properties:

- if f(z) = u + jv = analytic, u and v : satisfied Laplace equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

(u,v = harmonic func.)

- Let f(z),g(z) = analytic, f ± g ; f.g ; f/g (g≠0) = analytic .

- Polynomial function $f(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n$ (n = nonnegative integer) : analytic.

- Rational function f(z) = P(z)/Q(z) analytic in any domain S that contains no point z_0 for which $Q(z_0) = 0$.

7.7: Elementary functions :

1) The Exponential Function:

$$f(z) = e^{z} = e^{x+jy} = e^{x}\left(\cos y + j\sin y\right)$$

- Analytic and $\frac{d}{dz}(e^{z}) = e^{z}$

- **Modulus, argument and conjugate:**

$$|e^{z}| = e^{x}$$ $$\arg(e^{z}) = y + 2n\pi$$ $$(e^{z})^{*} = e^{z^{*}}$$

- **Periodic :** $e^{z+j2\pi} = e^{z}$

2) The Complex Logarithmic Function:

$$\boxed{f(z) = \ln z = \ln r + j[\theta + 2k\pi]} \quad (k = 0, \pm 1, \pm 2 \ldots)$$

- $\ln(z) =$ Multivalued function
- $\ln(z) = \ln r + j\theta \;\; (-\pi < \theta \leq \pi)$: called principal value

3) The Complex Power:

❖ **If α = complex number and $z \neq 0$, the complex power function is defined:**

$$\boxed{f(z) = z^{\alpha}}$$

- Calculate: $z^{\alpha} = e^{\alpha \ln z}$

- z^{α} = Multivalued function, $k = 0$:principal value

4) Trigonometric Functions :

$$\sin(z) = \frac{e^{jz} - e^{-jz}}{j2}$$

$$\cos(z) = \frac{e^{jz} + e^{-jz}}{2}$$

$$\sec(z) = \frac{1}{\cos(z)}$$

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

$$\cot(z) = \frac{\cos(z)}{\sin(z)}$$

$$\operatorname{cosec}(z) = \frac{1}{\sin(z)}$$

- **Following real vaiables, complex functions have inverses, written in terms of complex logarithm.**

- **Periodic with period = 2π .**

- $\sin(x + jy) = \sin x.\cosh y + j\cos x.\sinh y$
- $\cos(x + jy) = \cos x.\cosh y - j\sin x.\sinh y$

❖ Derivatives :

Derivatives of Complex Trigonometric Functions

$$\frac{d}{dz}\sin z = \cos z \qquad \frac{d}{dz}\cos z = -\sin z$$

$$\frac{d}{dz}\tan z = \sec^2 z \qquad \frac{d}{dz}\cot z = -\csc^2 z$$

$$\frac{d}{dz}\sec z = \sec z \tan z \qquad \frac{d}{dz}\csc z = -\csc z \cot z$$

5) The Complex Hyperbolic Functions :

$$\sinh(z) = \frac{e^{z} - e^{-z}}{2}$$

$$\cosh(z) = \frac{e^{z} + e^{-z}}{2}$$

$$\operatorname{sec}h(z) = \frac{1}{\cosh(z)}$$

$$\tanh(z) = \frac{\sinh(z)}{\cosh(z)}$$

$$\coth(z) = \frac{\cosh(z)}{\sinh(z)}$$

$$\operatorname{cosec}h(z) = \frac{1}{\sinh(z)}$$

$$\boxed{\sinh(x + jy) = \sinh x.\cos y + j\cosh.\sin y}$$

$$\boxed{\cosh(x + jy) = \cosh x.\cos y + j\sinh x.\sin y}$$

❖ Derivatives:

Derivatives of Complex Hyperbolic Functions

$$\frac{d}{dz}\sinh z = \cosh z \qquad \frac{d}{dz}\cosh z = \sinh z$$

$$\frac{d}{dz}\tanh z = \operatorname{sech}^2 z \qquad \frac{d}{dz}\coth z = -\operatorname{csch}^2 z$$

$$\frac{d}{dz}\operatorname{sech} z = -\operatorname{sech} z \tanh z \qquad \frac{d}{dz}\operatorname{csch} z = -\operatorname{csch} z \coth z$$