Chapter 8:

Complex Integrals

8.1: Complex Function w(t) :

❖ **Let w = f(z) defined in terms of one real variable t :**

$$w(t) = u(t) + jv(t)$$

$$\Rightarrow \int_a^b w(t)dt = \int_a^b u(t)dt + j\int_a^b v(t)dt$$

•Example1: Compute the integral $\int_0^2 (1 - jt)^2 dt$?

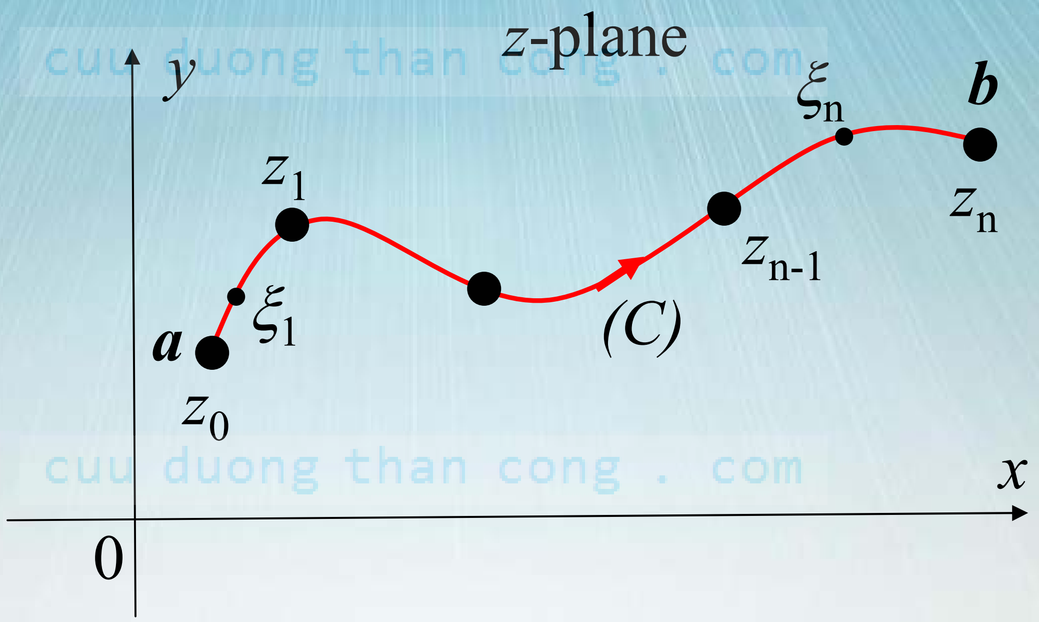
$$\int_0^2 (1 - t^2)dt = 2 - \tfrac{8}{3} \qquad -j\int_0^2 (2t)dt = -j4$$

$$\Rightarrow \int_0^2 (1 - jt)^2 dt = -\tfrac{2}{3} - j4$$

8.2: Complex Line Integrals :

- **Defined:**

$$\int_a^b f(z)dz = \int_C f(z)dz = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1})$$



• Connection between real & complex line integrals:

$$\int_C f(z)dz = \int_C (u + jv)(dx + jdy)$$

$$\int_C f(z)dz = \int_C (udx - vdy) + j\int_C (udy + vdx)$$

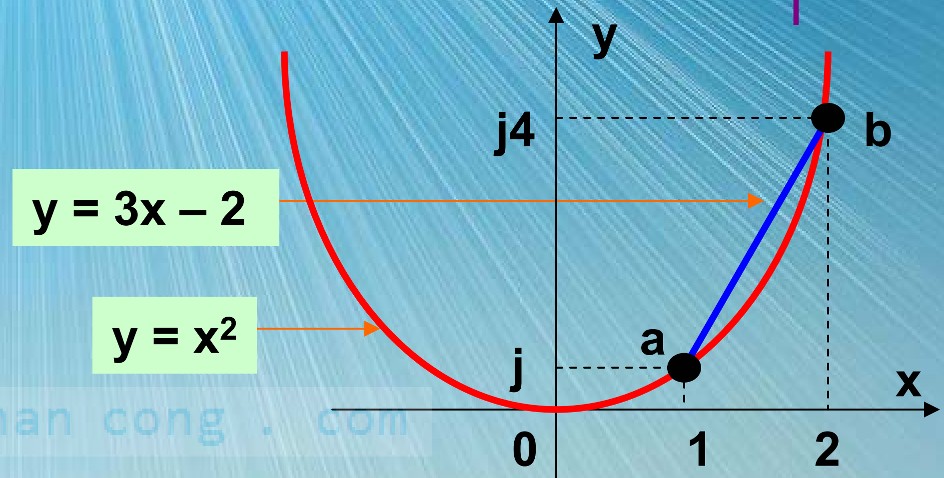
⇨ **Sum of two second line integrals .**

❖ Example1: Complex Line Integral

Compute $I = \int_a^b z^2 dz$?

a) Follow $y = x^2$?

b) Follow $y = 3x - 2$?



$$\int_a^b z^2 dz = \int_a^b [(x^2 - y^2)dx - 2xydy] + j\int_a^b [2xydx + (x^2 - y^2)dy]$$

a) $y = x^2 \rightarrow dy = 2xdx.$

$$I = \int_1^2 [x^2 - x^4 - 4x^4]dx + j\int_1^2 [2x^3 + (x^2 - x^4)2x]dx$$

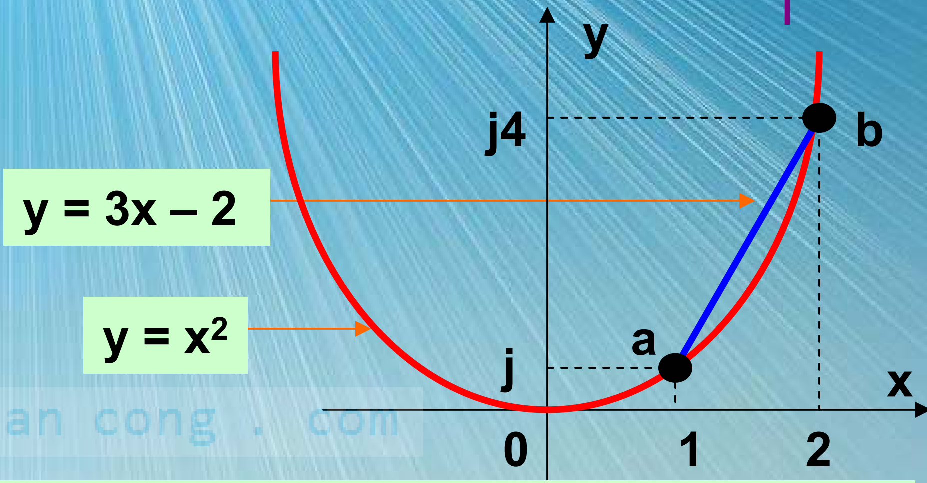
$$\boxed{I = [\tfrac{x^3}{3} - x^5]\Big|_1^2 + j[x^4 - \tfrac{x^6}{3}]\Big|_1^2 = -28.67 - j6}$$

❖Example1: Complex Line Integral

Compute $I = \int_a^b z^2 dz$?

a) Follow $y = x^2$?

b) Follow $y = 3x - 2$?



$$\int_a^b z^2 dz = \int_a^b [(x^2 - y^2)dx - 2xydy] + j\int_a^b [2xydx + (x^2 - y^2)dy]$$

b) $y = 3x - 2 \rightarrow dy = 3dx.$

$$I = \int_1^2 [x^2 - (3x-2)^2 - 6x(3x-2)]dx + j\int_1^2 [2x(3x-2) + (x^2 - (3x-2)^2)3]dx$$

$$\Rightarrow I = [-\frac{26x^3}{3} + 12x^2 - 4x]\Big|_1^2 + j[-6x^3 + 16x^2 - 12x]\Big|_1^2 = -28.67 - j6$$

Cauchy’s theorem :

Let f(z) analytic in S and its boundary C , then :

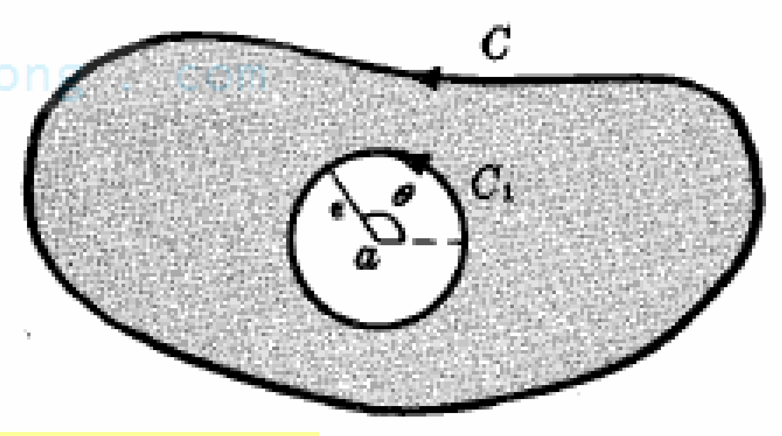
$$\oint_C f(z)dz = 0$$

8.3: Cauchy's Integral Formula :

- **Let f(z) analytic in S and its boundary C , a : any point inside C, then:**

$$f(a) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-a)}dz$$

(C: traversed in CCW)



❖**Proof:** $\oint_C \frac{f(z)}{(z-a)}dz = \oint_{C_1} \frac{f(z)}{(z-a)}dz$

Let $z - a = \varepsilon e^{j\theta} \Rightarrow dz = j\varepsilon e^{j\theta}d\theta$

$$\oint_{C_1} \frac{f(z)}{(z-a)}dz = \lim_{\varepsilon\to 0}\int_0^{2\pi} \frac{f(a+\varepsilon e^{j\theta})}{\varepsilon e^{j\theta}} j\varepsilon e^{j\theta}d\theta = jf(a)2\pi$$

$$\Rightarrow \quad f(a) = \frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-a)}dz$$

❖ And the second formula :

The n^{th} derivative of f(z) at z = a :

$$f^{(n)}(a) = \frac{n!}{2\pi j}\oint_C \frac{f(z)}{(z-a)^{n+1}}dz$$

(n = 1, 2, 3 …)

8.4: Poisson's Integral Formulae :

- Let C = cycle wih radius of R, using Cauchy formulae we have :

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2R.r\cos(\varphi - \theta) + r^2} u(R,\varphi)d\varphi$$