Chapter 9:

Series Complex Functions

9.1: Sequence :

a) Definition:

Sequence = an ordered set of numbers a_1, a_2, a_3, … so that $a_n = f(n)$. We can write $\{f(n)\}$ or $\{a_n\}$.

b) The Limit of a sequence:

Suppose the limit = L . Given a positive number ε, we can find a number N such that : $|a_n – L| < \varepsilon$ for all n > N.

And we can write : $\lim\limits_{n\to\infty}\{a_n\} = L$

L exist : convergent

L not exist : divergent

c) Sequence of Complex functions:

Complex Sequence = an ordered set of complex numbers a_1, a_2, a_3, … so that $a_n = f_n(z)$. We can write $\{f_n(z)\}$ or $\{a_n\}$.

If $|f_n(z) - f(z)| < \varepsilon$ for all $n > N$: sequence is convergent.

❖**Example1:** Is $a_n = 2/n^2 + j3$ convergent ? Find N ?

$$\lim_{n\to\infty}\{a_n\} = j3 \Rightarrow \left|\frac{2}{n^2}\right| < \varepsilon \Rightarrow N > \sqrt{\frac{2}{\varepsilon}}$$

9.2: Series Complex Function :

❖ **Series = By summing up the terms of a sequence.**

$$S_n = a_1(z) + a_2(z) + \ldots + a_n(z)$$

❖ **Infinite Series :** $\sum_{n=1}^{\infty} a_n(z)$

❖ **Convergent:** $\lim_{n\to\infty} S_n = S(z)$ **: Series is convergent**

Otherwise : Series is divergent

Convergent Tests :

i. Comparison Test: $\sum_{n=1}^{\infty} |a_n| :$ convergent

ii. Ratio Test: $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$\sum_{n=1}^{\infty} a_n :$ convergent

iii. Weierstrass M-test:

$|a_n| < M_n \ \& \ \sum_{n=1}^{\infty} M_n :$ convergent

uniformly

9.3: Power Series :

- Having the form:

$$\sum_{n=1}^{\infty} a_n (z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

(a_i: complex constant)

- Convergent: $\lim_{n\to\infty}\left|\frac{a_{n+1}(z-a)^{n+1}}{a_n z^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| |z-a| = L|z-a| < 1$

$\Rightarrow$ $\boxed{|z-a| < \frac{1}{L} = R}$: converges in a disc, center at a, radius of R.

9.4: Taylor Series :

❖ **Taylor series of f(z) , center at z = a, having the form:**

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \ldots$$

❖ **MacLaurin Series = Taylor series with center a = 0 :**

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(z)^n = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \ldots$$

❖ **A Taylor series converges in a disk, center at a, radius R.**

(Note: R = distance from a to nearest isolated singularity of f(z)).

❖ Some Special Series:

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \ldots$$

$$\frac{1}{1-z} = 1 + z + z^2 + z^3 + \ldots \ (\text{if } |z| < 1)$$

$$\frac{1}{1+z} = 1 - z + z^2 - z^3 + \ldots \ (\text{if } |z| < 1)$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$

9.5: Laurent Series :

❖ **If z = a is a singular point of f(z) [f(z) not analytic at z = a], we cannot expand f(z) in a power series center at z = a.**

Example : Cannot expand $f(z)=\frac{1}{1-z}$ in a power series at $z_0=1$.

❖ **Can represent f(z) in a series that contains both negative and positive integer power of (z – a) : Laurent series .**

Laurent Series :

❖ **Laurent series of f(z) for all z near a, having the form :**

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n = \ \ldots \ + \frac{a_{-2}}{(z-a)^2} + \frac{a_{-1}}{(z-a)} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots$$

where : $$a_n = \frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-a)^{n+1}} dz$$ ***(C: annulus enclosing the point a)***

❖ **The sum includes terms with negative powers: principal part .**

$$\frac{a_{-1}}{(z-a)} + \frac{a_{-2}}{(z-a)^2} + \cdots$$

The rest is analytic part.

❖Example1: Laurent Series

Expand $f(z) = \frac{\sin z}{z^4}$ in a Laurent series about $z_0 = 0$?

We have: $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$

$$f(z) = \frac{\sin z}{z^4} = \underbrace{\frac{1}{z^3} - \frac{1/3!}{z}}_{\text{(Principal part)}} + \underbrace{\frac{z}{5!} - \frac{z^3}{7!} + \ldots}_{\text{(Analytic part)}}$$

(Laurent Series)

❖ Find Laurent Series

Formula $a_n = \dfrac{1}{2\pi j}\oint_C \dfrac{f(z)}{(z-a)^{n+1}}dz$

is rarely use in practice.

- **Obtain Laurent Series by:**

i. Employing a known power series.

ii. Creation of geometric function .