Chapter 10:

Residue Theory

10.1: Singularities:

- Singular Point a of f(z) : f(z) not analytic at z = a but analytic in its neighborhood.

- Isolated Singularity of f(z) if z = a has a neighborhood without further singularities of f(z) .

Isolated Singularities classified:

- Using Laurent Series at isolated singulaity z = a:

$$\sum_{n=-\infty}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{m=1}^{\infty} \frac{b_m}{(z-a)^m}$$

(Analytic part) *(Principal part)*

i. If principal part is zero: z = a is removable singularity .

ii. If principal part contains m terms : z = a is pole of order m . If m = 1, z = a is simple pole.

iii. If principal part contains infinite terms : z = a is essential singularity.

Summary:

$z = z_0$	Laurent Series for $0 < \lvert z - z_0 \rvert < R$
Removable singularity	$a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Pole of order n	$\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-(n-1)}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$
Simple pole	$\frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$
Essential singularity	$\cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \cdots$

10.2: mth-order Zeros and Poles :

- Z = a is an mth-order zero if: $f(z) = (z-a)^m \varphi(z)$

($\varphi(z)$: analytic and $\varphi(a) \neq 0$)

- Z = a is an mth-order pole if: $f(z) = \dfrac{\varphi(z)}{(z-a)^m}$

($\varphi(z)$: analytic and $\varphi(a) \neq 0$)

10.3: Residue :

❖ **Residue of f(z) at z = a, written [Resf(z); a], defined:**

$$\mathrm{Res}\{f(z);a\} = a_{-1} \text{ of Laurent series}$$

•Example: Expand $f(z)=\dfrac{1}{(z-1)^2(z-3)}$ to Laurent Series in neiborhood of $z=1$:

a_{-1}

$$f(z)=\frac{-1/2}{(z-1)^2}+\frac{-1/4}{(z-1)}-\frac{1}{8}-\frac{z-1}{16}-\dots$$

❖ **Residue of f(z) at removable point is always 0 .**

10.4: Calculate Residue at a Pole :

a) Residue at a simple pole :

$$\operatorname{Res}\{f(z), z_0\} = \lim_{z \to z_0} \{f(z)(z - z_0)\}$$

- If $f(z) = \dfrac{P(z)}{Q(z)}$ then: $\operatorname{Res}\{f(z), z_0\} = \dfrac{P(z_0)}{Q'(z_0)}$

b) Residue at a Pole of order m :

$$\operatorname{Res}\{f(z), z_0\} = \frac{1}{(m-1)!} \lim_{z \to z_0} \left\{ \frac{d^{m-1}\left[f(z)(z-z_0)^m\right]}{dz^{m-1}} \right\}$$

10.5: Residue Theorem:

If f(z) = analytic on and within C, except at a finite number of singular points z_0, z_1, …, z_k within C, we have:

$$\oint_C f(z)dz = 2\pi j \sum_{k=1}^{n} \mathrm{Res}\{f(z), z_k\}$$