Chapter 11:

Application of Residue Theory

Applications of Residue Theorem

To evaluate real integrals

To determine Inversed Laplace Transform

11.1 Evaluation of definite Integrals :

a) Calculation $\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta$

(F(sin θ,cos θ) = rational function of sin θ and cos θ)

- **Let z = $e^{j\theta}$, we have sinθ = $(z - z^{-1})/2j$; cosθ = $(z + z^{-1})/2$ and dz = jz.dθ .**

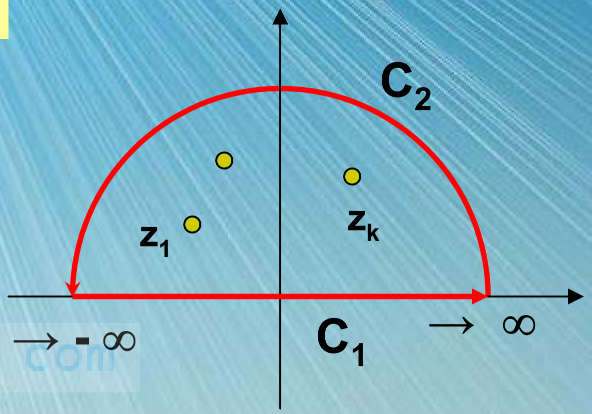
$$\int_0^{2\pi} F(\sin\theta, \cos\theta)d\theta = 2\pi j \sum_{k=1}^{n} \mathrm{Res}\{f(z), z_k\}$$

(z_k = poles in the unit cycle with centre at the origin)

b) Calculation $\int_{-\infty}^{\infty} f(x)dx$

(f(x) = p(x)/q(x) and Den ≥ Num + 2)

From $\oint_{C_1+C_2} f(z)dz = 2\pi j \sum_{k=1}^{n} \text{Res}\{f(z), z_k\}$



$$\int_{-\infty}^{\infty} f(x)dx = 2\pi j \sum_{k=1}^{n} \text{Res}\{f(z), z_k\}$$

(z_k = poles of f(z) in the upper half-plane)

- *If (f(x) = even function, we can evaluate :* $\int_{0}^{\infty} f(x)dx$

c) Calculation $\int_{-\infty}^{\infty} f(x)\cos(ax)dx$ and $\int_{-\infty}^{\infty} f(x)\sin(ax)dx$

(f(x) = p(x)/q(x) and Den ≥ Num + 1)

$$\int_{-\infty}^{\infty} f(x)\cos(ax)dx = -2\pi.\mathrm{Im}\left\{\sum_{k=1}^{n} \mathrm{Res}\{e^{iaz}f(z), z_k\}\right\}$$

$$\int_{-\infty}^{\infty} f(x)\sin(ax)dx = 2\pi.\mathrm{Re}\left\{\sum_{k=1}^{n} \mathrm{Res}\{e^{iaz}f(z), z_k\}\right\}$$

(z_k = poles of f(z) in the upper half-plane)

11.2 Inversed Laplace Transform :

- Formula: $f(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds$

- **Suppose that all singularities of F(s) are poles which lie to the left of the Re{s} = c.**

- **By using residue theorem:**

$$\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds = 2\pi j\sum \text{Res}\{F(s)e^{st}, \text{pole})$$

$$\boxed{f(t) = \sum \text{Res}\{F(s)e^{st}, \text{pole})}$$ ***(For t > 0)***

❖ Example1: Inversed Laplace Transform

$$\text{Find } f(t) \text{ if } F(s) = \frac{1}{s^2+\omega^2}$$

Simple poles = ± jω

$$\text{Residue}\{F(s)e^{st}, j\omega\} = \frac{e^{j\omega t}}{j2\omega}$$

$$\text{Residue}\{F(s)e^{st}, -j\omega\} = \frac{e^{-j\omega t}}{-j2\omega}$$

$$\int_{c-j\infty}^{c+j\infty} F(s)e^{st}ds = \sum(\text{enclosed Residues}) = \frac{e^{j\omega t}-e^{-j\omega t}}{j2\omega} = \frac{\sin(\omega t)}{\omega}$$

$$\boxed{\mathcal{L}^{-1}\{F(s)\} = \frac{\sin(\omega t)}{\omega}}$$