c SPKT TP.HCM
Khoa Khoa học ứng dụng

Bài tập Chương 9

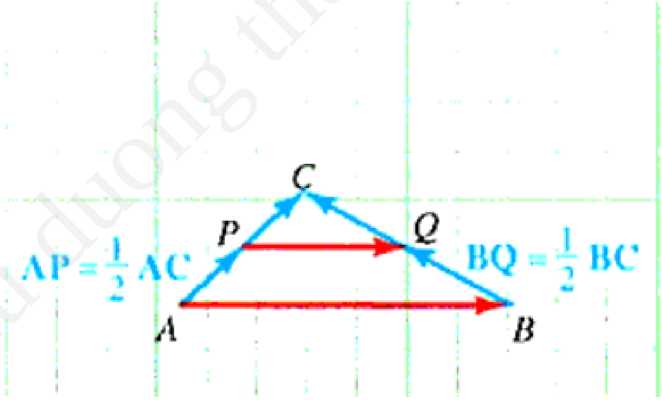
ng viên c Huy

9.1 Véc tơ trong R^2

Example 2 Vector proof of a geometric property

Show that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and has half its length.

Solution Consider ΔABC and let P and Q be the midpoints of sides $\overline{AC}$ and $\overline{BC}$, respectively, as shown in Figure 9.7.



9.1 Véc tơ trong R^2

Given that $\mathbf{AP} = \frac{1}{2}\mathbf{AC}$ and $\mathbf{BQ} = \frac{1}{2}\mathbf{BC}$ we must prove that $\mathbf{PQ}$ is parallel to $\mathbf{AB}$ and $\|\mathbf{PQ}\| = \frac{1}{2}\|\mathbf{AB}\|$, which means that we must establish the vector equation $\mathbf{PQ} = \frac{1}{2}\mathbf{AB}$. Toward this end, we begin by noting that $\mathbf{AB}$ can be expressed as the following vector sum:

$$
\begin{aligned}
\mathbf{AB} &= \mathbf{AP} + \mathbf{PQ} + \mathbf{QB} \\
&= \frac{1}{2}\mathbf{AC} + \mathbf{PQ} - \mathbf{BQ} && AP = \tfrac{1}{2}AC \text{ and } QB = -BQ \\
&= \frac{1}{2}(\mathbf{AB} + \mathbf{BC}) + \mathbf{PQ} - \frac{1}{2}\mathbf{BC} && AC = (AB + BC) \text{ and } BQ = \tfrac{1}{2}BC \\
&= \frac{1}{2}\mathbf{AB} + \frac{1}{2}\mathbf{BC} + \mathbf{PQ} - \frac{1}{2}\mathbf{BC} \\
&= \frac{1}{2}\mathbf{AB} + \mathbf{PQ}
\end{aligned}
$$

$$\frac{1}{2}\mathbf{AB} = \mathbf{PQ} \qquad \text{Subtract } \tfrac{1}{2}AB \text{ from both sides.}$$

9.2 Tọa độ và véc tơ trong R^3

Example 3 Equal vectors in component form

If $\mathbf{u} = \langle 8, -2\rangle$ and $\mathbf{v} = \langle -3, 5\rangle$, find s and t so that $s\mathbf{u} + t\mathbf{v} = \mathbf{w}$, where $\mathbf{w} = \langle 2, 8\rangle$.

Solution

$$\begin{aligned} s\mathbf{u} + t\mathbf{v} &= s\langle 8, -2\rangle + t\langle -3, 5\rangle \\ &= \langle 8s, -2s\rangle + \langle -3t, 5t\rangle \\ &= \langle 8s - 3t, -2s + 5t\rangle \end{aligned}$$

Thus, if $s\mathbf{u} + t\mathbf{v} = \mathbf{w}$, we see

$$\begin{cases} 8s - 3t = 2 \\ -2s + 5t = 8 \end{cases}$$

Solving this system we find $(s, t) = (1, 2)$.

9.2 Tọa độ và véc tơ trong R^3

Example 5 Finding a direction vector

Find a direction vector for the vector $\mathbf{v} = \langle 2, -3 \rangle$.

Solution The vector $\mathbf{v}$ has length (magnitude) $\|\mathbf{v}\| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$. Thus, the required direction vector is the unit vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle 2, -3 \rangle}{\sqrt{13}} = \frac{\sqrt{13}}{13}\langle 2, -3 \rangle = \left\langle \frac{2\sqrt{13}}{13}, \frac{-3\sqrt{13}}{13} \right\rangle$$

Example 7 Finding a direction vector

Find the unit vector that points in the direction of the vector $\mathbf{PQ}$ from $P(-1, 2, 5)$ to $Q(0, -3, 7)$.

Solution

$$\mathbf{PQ} = [0 - (-1)]\,\mathbf{i} + [-3 - 2]\mathbf{j} + [7 - 5]\,\mathbf{k} = \mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$$

$$\|\mathbf{PQ}\| = \sqrt{1^2 + (-5)^2 + 2^2} = \sqrt{30}$$

Thus,

$$\mathbf{u} = \frac{\mathbf{PQ}}{\|\mathbf{PQ}\|} = \frac{\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}}{\sqrt{30}} = \frac{1}{\sqrt{30}}\mathbf{i} - \frac{5}{\sqrt{30}}\mathbf{j} + \frac{2}{\sqrt{30}}\,\mathbf{k}$$

9.2 Tọa độ và véc tơ trong R^3

Example 5 Direction cosines and direction angles

Find the direction cosines and direction angles (to the nearest degree) of the vector $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$, and verify the formula

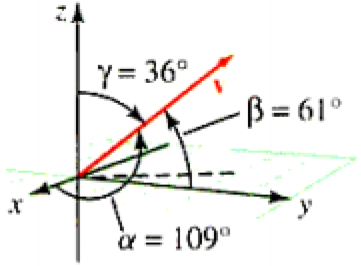
$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Solution We find $\|\mathbf{v}\| = \sqrt{(-2)^2 + 3^2 + 5^2} = \sqrt{38}$. Therefore,

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{-2}{\sqrt{38}} \approx -0.324428 \quad \alpha \approx \cos^{-1}(-0.324428) \approx 109^\circ$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{3}{\sqrt{38}} \approx 0.4866642 \quad \beta \approx \cos^{-1}(0.4866642) \approx 61^\circ$$

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{5}{\sqrt{38}} \approx 0.8111071 \quad \gamma \approx \cos^{-1}(0.8111071) \approx 36^\circ$$



and $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \left(\frac{-2}{\sqrt{38}}\right)^2 + \left(\frac{3}{\sqrt{38}}\right)^2 + \left(\frac{5}{\sqrt{38}}\right)^2 = 1$. These direction angles are shown in Figure 9.29.

9.2 Tọa độ và véc tơ trong R^3

Đồ thị trong không gian

Example 3 Graphing planes

Graph the planes defined by the given equations.

a. $x = 4$ **b.** $y + z = 5$ **c.** $x + 3y + 2z = 6$

Solution To graph a plane, find some ordered triples satisfying the equation. The best ones to use are often those that fall on a coordinate axis (the intercepts).

a. When two variables are missing, then the plane is parallel to one of the coordinate planes, as shown in Figure 9.18**a**.

b. When one of the variables is missing from an equation of a plane, then that plane is parallel to the axis corresponding the missing variable; in this case it is parallel to the x-axis. Draw the line $y + z = 5$ on the yz-plane, and then complete the plane, as shown in Figure 9.18**b**.

9.2 Tọa độ và véc tơ trong R^3

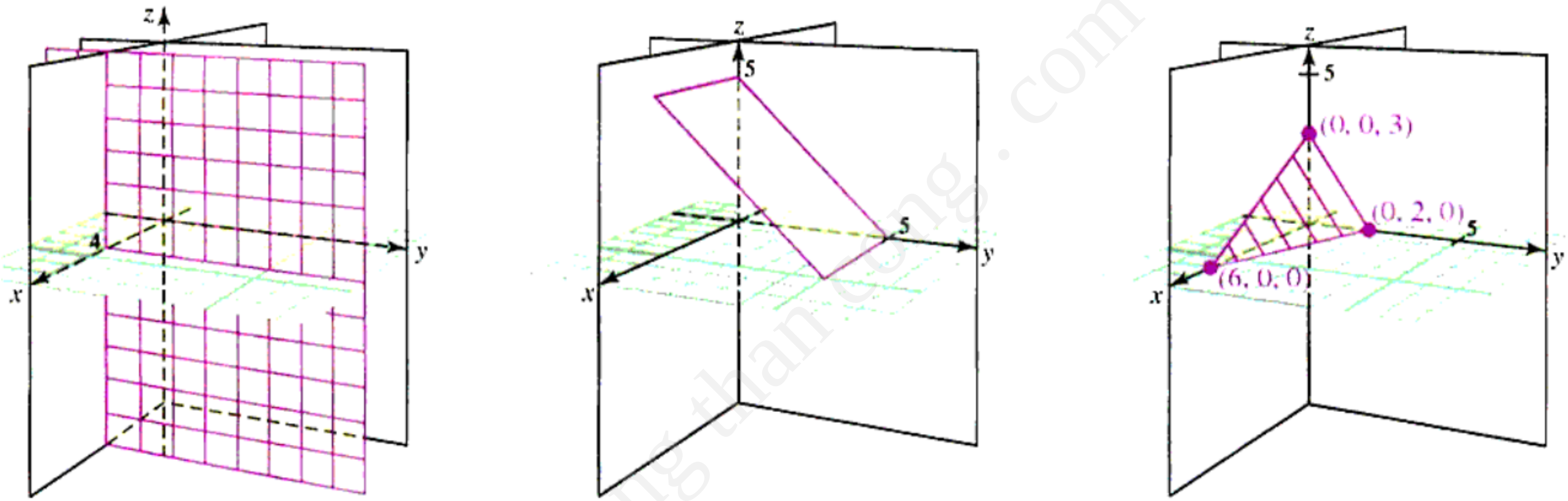


Figure 9.18 Graphs of planes

c. Let $x = 0$ and $y = 0$; then $z = 3$; plot the point $(0, 0, 3)$.
Let $x = 0$ and $z = 0$; then $y = 2$; plot the point $(0, 2, 0)$.
Let $y = 0$ and $z = 0$; then $x = 6$; plot the point $(6, 0, 0)$.
Use these points to draw the intersection lines (called **trace lines**) of the plane you are graphing with each of the coordinate planes. The result is shown in Figure 9.18**c**.

9.2 Tọa độ và véc tơ trong R^3

Example 4 Center and radius of a sphere from a given equation

Show that the graph of the equation $x^2 + y^2 + z^2 + 4x - 6y - 3 = 0$ is a sphere, and find its center and radius.

Solution By completing the square in both variables x and y, we have

$$(x^2 + 4x) + (y^2 - 6y) + z^2 = 3$$
$$(x^2 + 4x + 2^2) + (y^2 - 6y + (-3)^2) + z^2 = 3 + 4 + 9$$
$$(x + 2)^2 + (y - 3)^2 + z^2 = 16$$

9.2 Tọa độ và véc tơ trong R^3

Example 8 Parallel vectors

A vector **PQ** has initial point $P(1, 0, -3)$ and length 3. Find Q so that **PQ** is parallel to $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

Solution Let Q have coordinates (a_1, a_2, a_3). Then

$$\begin{aligned}\mathbf{PQ} &= [a_1 - 1]\mathbf{i} + [a_2 - 0]\mathbf{j} + [a_3 - (-3)]\mathbf{k} \\ &= (a_1 - 1)\mathbf{i} + a_2\mathbf{j} + (a_3 + 3)\mathbf{k}\end{aligned}$$

Because **PQ** is parallel to **v**, we have $\mathbf{PQ} = s\mathbf{v}$ for some scalar s, that is,

$$(a_1 - 1)\mathbf{i} + a_2\mathbf{j} + (a_3 + 3)\mathbf{k} = s(2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$$

Since the standard representation is unique, this implies that

$$\begin{array}{lll} a_1 - 1 = 2s & a_2 = -3s & a_3 + 3 = 6s \\ a_1 = 2s + 1 & & a_3 = 6s - 3 \end{array}$$

9.2 Tọa độ và véc tơ trong R^3

Because **PQ** has length 3, we have

$$\begin{aligned} 3 &= \sqrt{(a_1 - 1)^2 + a_2^2 + (a_3 + 3)^2} \\ &= \sqrt{[(2s + 1) - 1]^2 + (-3s)^2 + [(6s - 3) + 3]^2} \\ &= \sqrt{4s^2 + 9s^2 + 36s^2} \\ &= \sqrt{49s^2} \\ &= 7\,|s| \end{aligned}$$

Thus, $s = \pm\frac{3}{7}$ and for the two cases, we have

$s = \frac{3}{7}$: $\quad a_1 = 2\left(\frac{3}{7}\right) + 1 = \frac{13}{7} \quad a_2 = -3\left(\frac{3}{7}\right) = -\frac{9}{7} \quad a_3 = 6\left(\frac{3}{7}\right) - 3 = -\frac{3}{7}$

$s = -\frac{3}{7}$: $\quad a_1 = 2\left(-\frac{3}{7}\right) + 1 = \frac{1}{7} \quad a_2 = -3\left(-\frac{3}{7}\right) = \frac{9}{7} \quad a_3 = 6\left(-\frac{3}{7}\right) - 3 = -\frac{39}{7}$

There are two points that satisfy the conditions for the required terminal point Q: $\left(\frac{13}{7}, -\frac{9}{7}, -\frac{3}{7}\right)$ and $\left(\frac{1}{7}, \frac{9}{7}, -\frac{39}{7}\right)$.

9.3 Tích vô hướng của hai véc tơ

Example 3 Angle between two given vectors

Let $\triangle ABC$ be the triangle with vertices $A(1, 1, 8)$, $B(4, -3, -4)$, and $C(-3, 1, 5)$. Find the angle formed at A.

Solution Draw $\triangle ABC$ and label the angle formed at A as α as shown in Figure 9.27. The angle α is the angle between vectors **AB** and **AC**, where

$$\mathbf{AB} = (4-1)\mathbf{i} + (-3-1)\mathbf{j} + (-4-8)\mathbf{k} = 3\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$$

$$\mathbf{AC} = (-3-1)\mathbf{i} + (1-1)\mathbf{j} + (5-8)\mathbf{k} = -4\mathbf{i} - 3\mathbf{k}$$

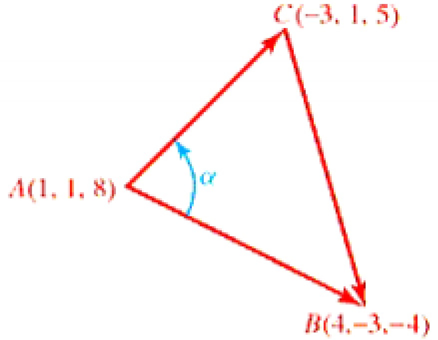


Figure 9.27 Angle in a triangle

Thus,

$$\cos\alpha = \frac{\mathbf{AB}\cdot\mathbf{AC}}{\|\mathbf{AB}\|\,\|\mathbf{AC}\|} = \frac{3(-4)+(-4)(0)+(-12)(-3)}{\sqrt{3^2+(-4)^2+(-12)^2}\,\sqrt{(-4)^2+(-3)^2}} = \frac{24}{\sqrt{169}\,\sqrt{25}} = \frac{24}{65}$$

and the required angle is $\alpha = \cos^{-1}\left(\frac{24}{65}\right) \approx 1.19$ (or about 68°).

9.3 Tích vô hướng của hai véc tơ

Example 6 Vector and scalar projections

Find the scalar and vector projections of $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$ onto $\mathbf{w} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$.

Solution We first find the vector projection:

$$\begin{aligned}\text{proj}_{\mathbf{w}}\mathbf{v} &= \left(\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} \\ &= \left(\frac{2(2)+3(-2)+5(-1)}{2^2+(-2)^2+(-1)^2}\right)(2\mathbf{i}-2\mathbf{j}-\mathbf{k}) \\ &= -\frac{7}{9}(2\mathbf{i}-2\mathbf{j}-\mathbf{k}) \\ &= -\frac{14}{9}\mathbf{i}+\frac{14}{9}\mathbf{j}+\frac{7}{9}\mathbf{k}\end{aligned}$$

To find the scalar projection, we can find the length of the vector projection, or we can use the scalar projection formula (which is usually easier than finding the length directly):

$$\begin{aligned}\text{comp}_{\mathbf{w}}\mathbf{v} &= \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|} \\ &= \frac{2(2)+3(-2)+5(-1)}{\sqrt{2^2+(-2)^2+(-1)^2}} \\ &= \frac{-7}{3}\end{aligned}$$

9.3 Tích có hướng của hai véc tơ

Example 1 Cross product

Find $\mathbf{v} \times \mathbf{w}$ where $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{w} = 7\mathbf{j} - 4\mathbf{k}$.

Solution

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

Do not forget minus here (j is negative position).

$$= [(-1)(-4) - 3(7)]\mathbf{i} - [2(-4) - 0(3)]\mathbf{j} + [2(7) - 0(-1)]\mathbf{k}$$

$$= -17\mathbf{i} + 8\mathbf{j} + 14\mathbf{k}$$

9.4 Tích có hướng của hai véc tơ

Example 4 Area of a triangle

Find the area of the triangle with vertices $P(-2,4,5)$, $Q(0,7,-4)$, and $R(-1,5,0)$.

Solution Draw this triangle as shown in Figure 9.40.

Then $\triangle PQR$ has half the area of the parallelogram determined by the vectors **PQ** and **PR**; that is, the triangle has area

$$A = \frac{1}{2}\|\mathbf{PQ} \times \mathbf{PR}\|$$

First find

$$\mathbf{PQ} = (0+2)\mathbf{i} + (7-4)\mathbf{j} + (-4-5)\mathbf{k} = 2\mathbf{i} + 3\mathbf{j} - 9\mathbf{k}$$

$$\mathbf{PR} = (-1+2)\mathbf{i} + (5-4)\mathbf{j} + (0-5)\mathbf{k} = \mathbf{i} + \mathbf{j} - 5\mathbf{k}$$

and compute the cross product:

$$\begin{aligned}\mathbf{PQ} \times \mathbf{PR} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & -9 \\ 1 & 1 & -5 \end{vmatrix} \\ &= (-15+9)\mathbf{i} - (-10+9)\mathbf{j} + (2-3)\mathbf{k} \\ &= -6\mathbf{i} + \mathbf{j} - \mathbf{k}\end{aligned}$$

Thus, the triangle has area

$$\begin{aligned}A &= \frac{1}{2}\|\mathbf{PQ} \times \mathbf{PR}\| \\ &= \frac{1}{2}\sqrt{(-6)^2 + 1^2 + (-1)^2} \\ &= \frac{1}{2}\sqrt{38}\end{aligned}$$

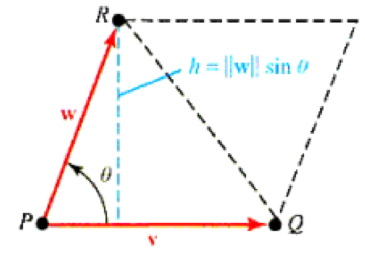


Figure 9.40 Area of a triangle

9.3 Tích có hướng của hai véc tơ

Example 5 Volume of a parallelepiped

Find the volume of the parallelepiped determined by the vectors $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, $\mathbf{v} = -4\mathbf{i} + 7\mathbf{j} - 11\mathbf{k}$, and $\mathbf{w} = 5\mathbf{i} + 9\mathbf{j} - \mathbf{k}$.

Solution We first find the cross product.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 3 \\ -4 & 7 & -11 \end{vmatrix} \\ &= (22 - 21)\mathbf{i} - (-11 + 12)\mathbf{j} + (7 - 8)\mathbf{k} \\ &= \mathbf{i} - \mathbf{j} - \mathbf{k} \end{aligned}$$

Thus,

$$\begin{aligned} V &= |(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}| \\ &= |(\mathbf{i} - \mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} + 9\mathbf{j} - \mathbf{k})| \\ &= |5 - 9 + 1| \\ &= 3 \end{aligned}$$

The volume of the parallelepiped is 3 cubic units.

9.5 Đường trong R^3

PARAMETRIC FORM OF A LINE IN $\mathbb{R}^3$ If L is a line that contains the point (x_0, y_0, z_0) and is aligned with the vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, then the point (x, y, z) is on L if and only if its coordinates satisfy

$$x - x_0 = tA \qquad y - y_0 = tB \qquad z - z_0 = tC$$

for some number t.

Example 1 Parametric equations of a line in space

Find the parametric equations for the line that contains the point $(3, 1, 4)$ and is aligned with the vector $\mathbf{v} = -\mathbf{i} + \mathbf{j} - 2\mathbf{k}$. Find where this line passes through the coordinate planes and sketch the line.

Solution The direction numbers are $[-1, 1, -2]$ and $x_0 = 3$, $y_0 = 1$, $z_0 = 4$, so the line has the parametric form

$$x - 3 = -t \qquad y - 1 = t \qquad z - 4 = -2t$$
$$x = 3 - t \qquad y = 1 + t \qquad z = 4 - 2t$$

This line will intersect the xy-plane when $z = 0$; solve

$$0 = 4 - 2t \qquad \text{implies} \qquad t = 2$$

If $t = 2$, then $x = 3 - 2 = 1$ and $y = 1 + 2 = 3$. This is the point $(1, 3, 0)$. Similarly, the line intersects the xz-plane at $(4, 0, 6)$ and the yz-plane at $(0, 4, -2)$. Plot these points and draw the line, as shown in Figure 9.48.

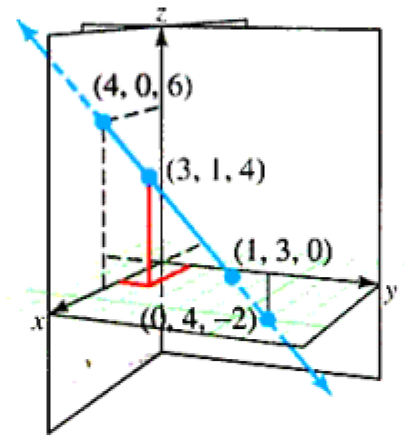


Figure 9.48 Graph of a line in space

9.5 Đường trong R^3

SYMMETRIC FORM OF A LINE IN $\mathbb{R}^3$ If L is a line that contains the point (x_0, y_0, z_0) and is aligned with the vector $\mathbf{v} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$, ($A$, B, and C nonzero numbers) then the point (x, y, z) is L if and only if its coordinates satisfy

$$\frac{x - x_0}{A} = \frac{y - y_0}{B} = \frac{z - z_0}{C}$$

Example 2 Symmetric form of the equation of a line in space

Find symmetric equations for the line L through the points $P(-1, 3, 7)$ and $Q(4, 2, -1)$. Find the points of intersection with the coordinate planes and sketch the line.

Solution The required line passes through P and is aligned with the vector

$$\mathbf{PQ} = [4 - (-1)]\mathbf{i} + [2 - 3]\mathbf{j} + [-1 - 7]\mathbf{k} = 5\mathbf{i} - \mathbf{j} - 8\mathbf{k}$$

Thus, the direction numbers of the line are $[5, -1, -8]$, and we can choose either P or Q as (x_0, y_0, z_0). Choosing P, we obtain:

$$\frac{x + 1}{5} = \frac{y - 3}{-1} = \frac{z - 7}{-8}$$

Next, we find points of intersection with the coordinate planes:

xy-plane: $z = 0$, so $\frac{x+1}{5} = \frac{7}{8}$ implies $x = \frac{27}{8}$ and $\frac{y-3}{-1} = \frac{7}{8}$ implies $y = \frac{17}{8}$.

The point of intersection of the line with the xy-plane is $(\frac{27}{8}, \frac{17}{8}, 0)$. Similarly, the other intersections are xz-plane: $(14, 0, -17)$; yz-plane: $(0, \frac{14}{5}, \frac{27}{5})$. The graph is shown in Figure 9.49.

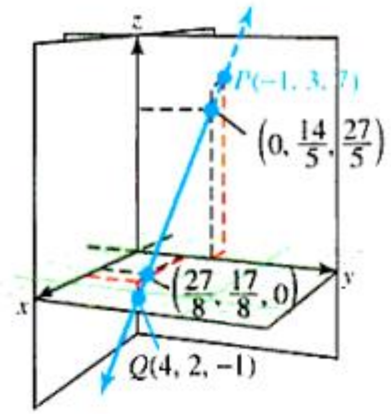


Figure 9.49 Interactive Graph of $\frac{x+1}{5} = \frac{y-3}{-1} = \frac{z-7}{-8}$

9.5 Đường trong R^3

Example 3 Skew lines in space

Determine whether the following pair of lines intersect, are parallel, or are skew.

$$L_1: \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{4} \qquad \text{and} \qquad L_2: \frac{x+2}{4} = \frac{y}{-3} = \frac{z+1}{1}$$

Solution Note that L_1 has direction numbers $[2, 1, 4]$ (that is, L_1 is aligned with $2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$) and L_2 has direction numbers $[4, -3, 1]$. If we solve $\langle 2, 1, 4\rangle = t\langle 4, -3, 1\rangle$ for t, we find no possible solution for any value of t. This implies that the lines are not parallel. (See Figure 9.51.)

Next, we determine whether the lines intersect or are skew. Note that $S(1, -1, 2)$ lies on L_1 and $T(-2, 0, -1)$ lies on L_2. The lines intersect if and only if there is a point P that lies on both lines. To determine this, we write the equations of the lines in parametric form. We use a different parameter for each line so that points on one line do not depend on the value of the parameter of the other line:

L_1:	$x = 1 + 2s$	$y = -1 + s$	$z = 2 + 4s$
L_2:	$x = -2 + 4t$	$y = -3t$	$z = -1 + t$
Thus:	$x = 1 + 2s = -2 + 4t$	$y = -1 + s = -3t$	$z = 2 + 4s = -1 + t$
or:	$2s - 4t = -3$	$s + 3t = 1$	$4s - t = -3$

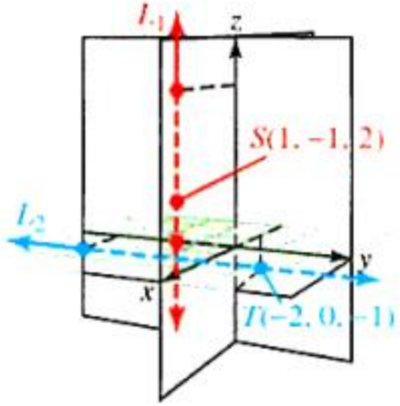


Figure 9.51 Interactive Graphs of L_1 and L_2

9.5 Đường trong R^3

This is equivalent to the system of linear equations

$$\begin{cases} 2s - 4t = -3 \\ s + 3t = 1 \\ 4s - t = -3 \end{cases}$$

Any solution of this system must correspond to a point of intersection of L_1 and L_2, and if no solution exists, then L_1 and L_2 are skew. Because this is a system of three equations with two unknowns, we first solve the first two equations simultaneously to find $s = -\frac{1}{2}, t = \frac{1}{2}$. Because $s = -\frac{1}{2}$ and $t = \frac{1}{2}$ do not satisfy the third equation, it follows that L_1 and L_2 do not intersect, so they must be skew.

9.5 Đường trong R^3

Example 4 Intersecting lines

Show that the lines

$$L_1: \frac{x-1}{2} = \frac{y+1}{1} = \frac{z-2}{4} \quad \text{and} \quad L_2: \frac{x+2}{4} = \frac{y}{-3} = \frac{z-\frac{1}{2}}{-1}$$

intersect and find the point of intersection. (See Figure 9.52.)

Solution L_1 has direction numbers $[2, 1, 4]$ and L_2 has direction numbers $[4, -3, -1]$. Because there is no t for which $[2, 1, 4] = t[4, -3, -1]$, the lines are not parallel. Express the lines in parametric form:

$$L_1: \quad x = 1 + 2s \quad y = -1 + s \quad z = 2 + 4s$$

$$L_2: \quad x = -2 + 4t \quad y = -3t \quad z = \frac{1}{2} - t$$

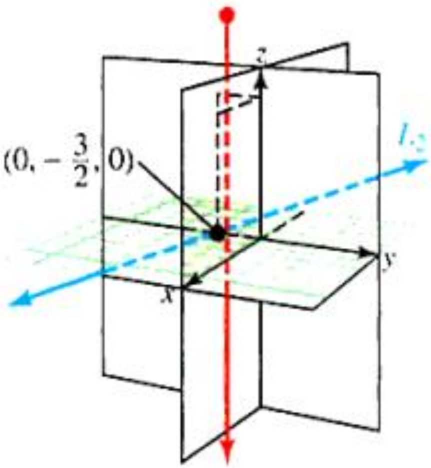
$$\text{Thus:} \quad x = 1 + 2s = -2 + 4t \quad y = -1 + s = -3t \quad z = 2 + 4s = \frac{1}{2} - t$$

$$\text{or:} \quad 2s - 4t = -3 \quad s + 3t = 1 \quad 4s + t = -\frac{3}{2}$$

Solving the first two equations simultaneously, we find $s = -\frac{1}{2}, t = \frac{1}{2}$. This solution satisfies the third equation, namely,

$$4\left(-\frac{1}{2}\right) + \frac{1}{2} = -\frac{3}{2}$$

9.5 Đường trong R^3



To find the coordinates of the point of intersection, substitute $s = -\frac{1}{2}$ into the parametric-form equations for L_1 (or substitute $t = \frac{1}{2}$ into L_2) to obtain

$$x_0 = 1 + 2\left(-\frac{1}{2}\right) = 0$$
$$y_0 = -1 + \left(-\frac{1}{2}\right) = -\frac{3}{2}$$
$$z_0 = 2 + 4\left(-\frac{1}{2}\right) = 0$$

Thus, the lines intersect at $P(0, -\frac{3}{2}, 0)$.

9.5 Đường trong R^3

PARAMETRIC REPRESENTATION Let f_1, f_2, and f_3 be continuous functions of t on an interval I; then the equations

$$x = f_1(t), \quad y = f_2(t), \quad z = f_3(t)$$

are called **parametric equations** with **parameter** t. As t varies over the **parametric set** I, the points

$$(x, y, z) = (f_1(t), f_2(t), f_3(t))$$

trace out a **parametric curve** in $\mathbb{R}^3$. If $z = f_3(t) = 0$, then the curve is in the xy-plane and we say the parametric curve is in $\mathbb{R}^2$.

9.5 Đường trong R^3

Example 5 Sketching the path of a parametric curve

Sketch the path of the curve $x = t^2 - 9$, $y = \frac{1}{3}t$ for $-3 \le t \le 2$.

Solution Values of x and y corresponding to various choices of the parameter t are shown in the following table:

t	x	y	
-3	0	-1	(Starting or **initial** point)
-2	-5	$-\frac{2}{3}$	
-1	-8	$-\frac{1}{3}$	
0	-9	0	
1	-8	$\frac{1}{3}$	
2	-5	$\frac{2}{3}$	(Ending or **terminal** point)

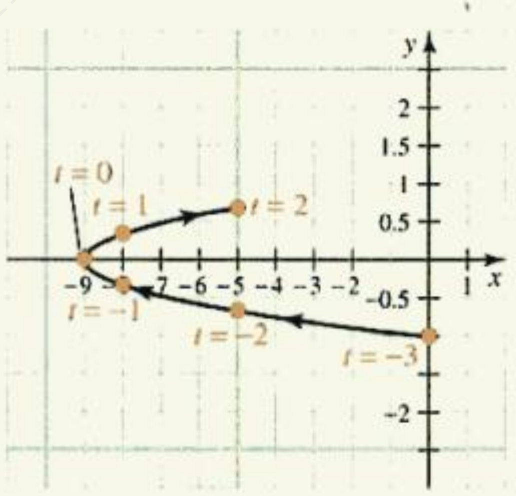


Figure 9.53 **Interactive** Graph of spiral with restricted domain

The graph is shown in Figure 9.53. Note how the arrows show the orientation as t increases from –3 to 2.

9.5 Đường trong R^3

Example 6 Sketching the path by eliminating the parameter

Describe the path $x = \sin \pi t$, $y = \cos 2\pi t$ for $0 \le t \le 0.5$.

Solution Using a double angle identity, we find

$$\cos 2\pi t = 1 - 2\sin^2 \pi t$$

so that

$$y = 1 - 2x^2$$

We recognize this as a Cartesian equation for a parabola. Because $y' = -4x$, we can find the critical number $x = 0$, which locates the vertex of the parabola at $(0, 1)$. The parabola is the curve shown in color as the dashed curve in Figure 9.54.

Because t is restricted to the interval $0 \le t \le 0.5$, the parametric representation involves only part of the right side of the parabola $y = 1 - 2x^2$. The curve is oriented from the point $(0, 1)$, where $t = 0$, to the point $(1, -1)$, where $t = 0.5$, and is the portion of the parabola shown in black in Figure 9.54.

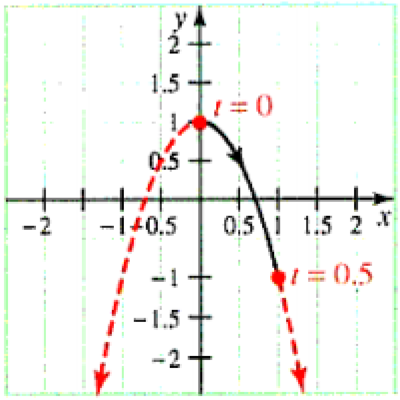


Figure 9.54 Interactive Graph of parabola with restricted domain

When it is difficult to eliminate the parameter from a given parametric representation, we can sometimes get a good picture of the parametric curve by plotting points.

9.5 Đường trong R^3

Example 7 Describing a spiraling path

Discuss the path of the curve described by the parametric equations

$$x = e^{-t}\cos t, \quad y = e^{-t}\sin t \qquad \text{for } t \geq 0$$

Solution We have no convenient way of eliminating the parameter so we write out a table of values (x, y) that correspond to various values of t. The curve is obtained by plotting these points in a Cartesian plane and passing a smooth curve through the plotted points, as shown in Figure 9.55.

t	x	y
0	1	0
$\frac{\pi}{4}$	0.32	0.32
1	0.20	0.31
$\frac{\pi}{2}$	0	0.21
2	−0.06	0.12
π	−0.04	0
$\frac{3\pi}{2}$	0	−0.01
2π	0.00	0

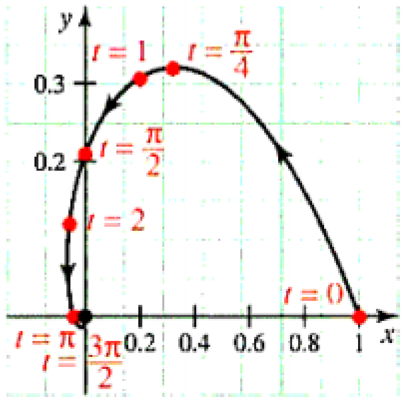


Figure 9.55 **Interactive** Graph of spiral with restricted domain

Đường trong R^3

Example 8 Parameterizing two curves

In each of the following cases, parameterize the given curve:

a. $y = 9x^2$ **b.** $r = 5\cos^3\theta$ in polar coordinates

Solution

a. The usual parameterization for a parabola is to let the parameter t be the variable that is squared: $x = t$, $y = 9t^2$. However, another parameterization is to let $t = 3x$ so that $x = \frac{1}{3}t$ and $y = t^2$.

b. In polar coordinates we have $x = r\cos\theta$, $y = r\sin\theta$, so we can parameterize x and y in terms of the parameter θ:

$$\begin{aligned} x &= r\cos\theta & y &= r\sin\theta \\ &= \left(5\cos^3\theta\right)\cos\theta & &= \left(5\cos^3\theta\right)\sin\theta \\ &= 5\cos^4\theta & &= 5\cos^3\theta\sin\theta \end{aligned}$$