UNIVERSITY OF TECHNOLOGY AND EDUCATION
FACULTY OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHATRONICS

LINEAR ALGEBRA

AMME_131529

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Matrices, Vectors: Addition and Scalar Multiplication

$$1.3\,p - 2.0q + r = 7$$

$$3.7\,p + 4.8q - 7r = 3$$

$$4.1\,p + 3.8q + 12r = -6$$

Hệ phương trình

$$\begin{pmatrix} 1.3 & -2.0 & 1 \\ 3.7 & 4.8 & -7 \\ 4.1 & 3.8 & 12 \end{pmatrix}$$

Ma trận 3 hàng 3 cột (9 phần tử)

Hàng (ngang)

Cột (dọc)

Matrices, Vectors: Addition and Scalar Multiplication

$$\begin{aligned} 4x_1 + 6x_2 + 9x_3 &= 6 \\ 6x_1 \qquad\quad - 2x_3 &= 20 \\ 5x_1 - 8x_2 + x_3 &= 10 \end{aligned}$$

Hệ phương trình

Tìm ma trận

Matrices, Vectors: Addition and Scalar Multiplication

$$\begin{bmatrix} 0.3 & 1 & -5 \\ 0 & -0.2 & 16 \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

$$\begin{bmatrix} e^{-x} & 2x^2 \\ e^{6x} & 4x \end{bmatrix}, \quad [a_1 \quad a_2 \quad a_3], \quad \begin{bmatrix} 4 \\ \frac{1}{2} \end{bmatrix}$$

Xác định kích thước (số hàng và cột của mỗi ma trận)

Matrices, Vectors: Addition and Scalar Multiplication

Matrices

$$\mathbf{A} = [a_{jk}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Matrices, Vectors: Addition and Scalar Multiplication

Vectors

$$\mathbf{a} = [a_1 \quad a_2 \quad \cdots \quad a_n]$$

$$\mathbf{a} = [-2 \quad 5 \quad 0.8 \quad 0 \quad 1].$$

Equality of Matrices

Two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ are **equal**, written $\mathbf{A} = \mathbf{B}$, if and only if they have the same size and the corresponding entries are equal, that is, $a_{11} = b_{11}$, $a_{12} = b_{12}$, and so on. Matrices that are not equal are called **different**. Thus, matrices of different sizes are always different.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$$

$$\mathbf{A} = \mathbf{B} \qquad \text{if and only if} \qquad \begin{matrix} a_{11} = 4, & a_{12} = 0, \\ a_{21} = 3, & a_{22} = -1 \end{matrix}$$

Addition of Matrices

The **sum** of two matrices $\mathbf{A} = [a_{jk}]$ and $\mathbf{B} = [b_{jk}]$ ***of the same size*** is written $\mathbf{A} + \mathbf{B}$ and has the entries $a_{jk} + b_{jk}$ obtained by adding the corresponding entries of $\mathbf{A}$ and $\mathbf{B}$. Matrices of different sizes cannot be added.

$$\mathbf{A} = \begin{bmatrix} -4 & 6 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 5 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

$$\text{then} \quad \mathbf{A} + \mathbf{B} = \begin{bmatrix} 1 & 5 & 3 \\ 3 & 2 & 2 \end{bmatrix}.$$

Matrices, Vectors: Addition and Scalar Multiplication

Addition of matrices

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Addition of matrices

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} \text{ and } \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} + \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 2+(-3) & -1+0 \\ -7+7 & 4+(-4) \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Addition of matrices

$$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Addition of matrices

$$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3+2 & 1+7 & -4+(-5) \\ 4+(-2) & 3+1 & 1+0 \\ 1+6 & 4+3 & -3+4 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 8 & -9 \\ 2 & 4 & 1 \\ 7 & 7 & 1 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Subtraction of matrices

$$\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} \text{ from } \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$$

Subtraction of matrices

$$\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} \text{ from } \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} - \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 2-(-3) & -1-0 \\ -7-7 & 4-(-4) \end{pmatrix}$$

$$= \begin{pmatrix} 5 & -1 \\ -14 & 8 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Subtraction of matrices

$$\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix} \text{ from } \begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Subtraction of matrices

$$\begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix} \text{ from } \begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 1 & -4 \\ 4 & 3 & 1 \\ 1 & 4 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 7 & -5 \\ -2 & 1 & 0 \\ 6 & 3 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 3-2 & 1-7 & -4-(-5) \\ 4-(-2) & 3-1 & 1-0 \\ 1-6 & 4-3 & -3-4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -6 & 1 \\ 6 & 2 & 1 \\ -5 & 1 & -7 \end{pmatrix}$$

Scalar Multiplication (Multiplication by a Number)

The **product** of any $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ and any **scalar** c (number c) is written $c\mathbf{A}$ and is the $m \times n$ matrix $c\mathbf{A} = [ca_{jk}]$ obtained by multiplying each entry of $\mathbf{A}$ by c.

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 2.7 & -1.8 \\ 0 & 0.9 \\ 9.0 & -4.5 \end{bmatrix}, \quad \text{then} \quad -\mathbf{A} = \begin{bmatrix} -2.7 & 1.8 \\ 0 & -0.9 \\ -9.0 & 4.5 \end{bmatrix},$$

$$\frac{10}{9}\mathbf{A} = \begin{bmatrix} 3 & -2 \\ 0 & 1 \\ 10 & -5 \end{bmatrix}, \quad 0\mathbf{A} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Multiplication

If $A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$,

$B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find

$2A - 3B + 4C$

Matrices, Vectors: Addition and Scalar Multiplication

Multiplication

If $A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$,

$B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find

$2A - 3B + 4C$

$$2A = 2\begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix} = \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix}$$

$$3B = 3\begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix} = \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix}$$

$$4C = 4\begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

Multiplication of a Matrix by a Matrix

The **product** $\mathbf{C} = \mathbf{AB}$ (in this order) of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ times an $r \times p$ matrix $\mathbf{B} = [b_{jk}]$ is defined if and only if $r = n$ and is then the $m \times p$ matrix $\mathbf{C} = [c_{jk}]$ with entries

$$c_{jk} = \sum_{l=1}^{n} a_{jl}b_{lk} = a_{j1}b_{1k} + a_{j2}b_{2k} + \cdots + a_{jn}b_{nk} \qquad \begin{array}{l} j = 1, \cdots, m \\ k = 1, \cdots, p. \end{array}$$

$$\begin{array}{cccc} \mathbf{A} & \mathbf{B} & = & \mathbf{C} \\ [m \times n] & [n \times p] & = & [m \times p] \end{array}$$

Matrix Multiplication Is Not Commutative

$$\mathbf{AB} \neq \mathbf{BA}$$

$$(k\mathbf{A})\mathbf{B} = k(\mathbf{AB}) = \mathbf{A}(k\mathbf{B})$$

$$\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$$

associative law

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

$$\mathbf{C}(\mathbf{A} + \mathbf{B}) = \mathbf{CA} + \mathbf{CB}$$

distributive laws

Matrices, Vectors: Addition and Scalar Multiplication

Multiplication

If $A = \begin{pmatrix} -3 & 0 \\ 7 & -4 \end{pmatrix}$,

$B = \begin{pmatrix} 2 & -1 \\ -7 & 4 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \\ -2 & -4 \end{pmatrix}$ find

$2A - 3B + 4C$

Hence $2A - 3B + 4C$

$$= \begin{pmatrix} -6 & 0 \\ 14 & -8 \end{pmatrix} - \begin{pmatrix} 6 & -3 \\ -21 & 12 \end{pmatrix} + \begin{pmatrix} 4 & 0 \\ -8 & -16 \end{pmatrix}$$

$$= \begin{pmatrix} -6 - 6 + 4 & 0 - (-3) + 0 \\ 14 - (-21) + (-8) & -8 - 12 + (-16) \end{pmatrix}$$

$$= \begin{pmatrix} -8 & 3 \\ 27 & -36 \end{pmatrix}$$

Matrices, Vectors:
Addition and Scalar Multiplication

Multiplication

Simplify

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Multiplication

Simplify

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 5 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} (3 \times 2) + (4 \times 5) + (0 \times (-1)) \\ (-2 \times 2) + (6 \times 5) + (-3 \times (-1)) \\ (7 \times 2) + (-4 \times 5) + (1 \times (-1)) \end{pmatrix}$$

$$= \begin{pmatrix} 26 \\ 29 \\ -7 \end{pmatrix}$$

Matrices, Vectors: Addition and Scalar Multiplication

Multiplication

If $A = \begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix}$ and

$B = \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix}$, find $A \times B$

Matrices, Vectors: Addition and Scalar Multiplication

Multiplication

$$\begin{pmatrix} 3 & 4 & 0 \\ -2 & 6 & -3 \\ 7 & -4 & 1 \end{pmatrix} \times \begin{pmatrix} 2 & -5 \\ 5 & -6 \\ -1 & -7 \end{pmatrix} = \begin{pmatrix} [(3 \times 2) & [(3 \times (-5)) \\ +(4 \times 5) & +(4 \times (-6)) \\ +(0 \times (-1))] & +(0 \times (-7))] \\ [(-2 \times 2) & [(-2 \times (-5)) \\ +(6 \times 5) & +(6 \times (-6)) \\ +(-3 \times (-1))] & +(-3 \times (-7))] \\ [(7 \times 2) & [(7 \times (-5)) \\ +(-4 \times 5) & +(-4 \times (-6)) \\ +(1 \times (-1))] & +(1 \times (-7))] \end{pmatrix}$$

$$= \begin{pmatrix} 26 & -39 \\ 29 & -5 \\ -7 & -18 \end{pmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$\mathbf{AB} = \begin{bmatrix} 3 & 5 & -1 \\ 4 & 0 & 2 \\ -6 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 & 3 & 1 \\ 5 & 0 & 7 & 8 \\ 9 & -4 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -2 & 43 & 42 \\ 26 & -16 & 14 & 6 \\ -9 & 4 & -37 & -28 \end{bmatrix}$$

$$c_{23} = 4 \cdot 3 + 0 \cdot 7 + 2 \cdot 1 = 14$$

Matrices, Vectors: Addition and Scalar Multiplication

A **unit matrix, *I***, is one in which all elements of the leading diagonal (\) have a value of 1 and all other elements have a value of 0.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

unit matrix: Ma trận đơn vị

Ma trận đơn vị là ma trận có các phần tử trên đường chéo chính có giá trị 1, còn các phần tử khác có giá trị 0

Transposition (Chuyển vị)

We obtain the transpose of a matrix by writing its rows as columns (or equivalently its columns as rows)

$$\text{If} \quad \mathbf{A} = \begin{bmatrix} 5 & -8 & 1 \\ 4 & 0 & 0 \end{bmatrix} \text{ then } \quad \mathbf{A}^T = \begin{bmatrix} 5 & 4 \\ -8 & 0 \\ 1 & 0 \end{bmatrix}$$

Chúng ta đạt được ma trận chuyển vị bằng cách chuyển hàng thành cột (cột thành hàng)

Transposition of Matrices and Vectors

The transpose of an $m \times n$ matrix $\mathbf{A} = [a_{jk}]$ is the $n \times m$ matrix $\mathbf{A}^\mathsf{T}$ (read *A transpose*) that has the first *row* of $\mathbf{A}$ as its first *column*, the second *row* of $\mathbf{A}$ as its second *column*, and so on. Thus the transpose of $\mathbf{A}$ in (2) is $\mathbf{A}^\mathsf{T} = [a_{kj}]$, written out

$$\mathbf{A}^\mathsf{T} = [a_{kj}] = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \cdot & \cdot & \cdots & \cdot \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$(\mathbf{A} + \mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$$

$$(c\mathbf{A})^T = c\mathbf{A}^T$$

$$(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$$

Symmetric Matrix

$$\mathbf{A}^\mathsf{T} = \mathbf{A} \qquad (\text{thus } a_{kj} = a_{jk}),$$

Skew-Symmetric Matrices

$$\mathbf{A}^\mathsf{T} = -\mathbf{A} \qquad (\text{thus } a_{kj} = -a_{jk}, \text{ hence } a_{jj} = 0)$$

Symmetric Matrix

$$\mathbf{A} = \begin{bmatrix} 20 & 120 & 200 \\ 120 & 10 & 150 \\ 200 & 150 & 30 \end{bmatrix}$$

Skew-Symmetric Matrices

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix}$$

Triangular Matrices

Upper triangular matrices

$$\begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 4 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 6 \end{bmatrix}$$

Lower triangular matrices

$$\begin{bmatrix} 2 & 0 & 0 \\ 8 & -1 & 0 \\ 7 & 6 & 8 \end{bmatrix}, \quad \begin{bmatrix} 3 & 0 & 0 & 0 \\ 9 & -3 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 9 & 3 & 6 \end{bmatrix}$$

Diagonal Matrices. These are square matrices that can have nonzero entries only on the main diagonal

Scalar matrix

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A}$$

Unit matrix (or identity matrix)

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

Diagonal Matrices.

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

Scalar matrix

$$\mathbf{AS} = \mathbf{SA} = c\mathbf{A}$$

$$\mathbf{S} = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & c \end{bmatrix}$$

Unit matrix (or identity matrix)

$$\mathbf{AI} = \mathbf{IA} = \mathbf{A}$$

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 261 (ex 8-16) Trang 271 (ex 11-20)

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 261 (ex 8-16) Trang 271 (ex 11-20)

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 261 (ex 8-16) Trang 271 (ex 11-20)

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 287 (ex 1-9) (ex 17-26)
- Sách Advanced Engineering Mathematics 10th Edition, Trang 300 (ex 7-15), (ex 17-19), (ex 21-25)
- Sách Advanced Engineering Mathematics 10th Edition, Trang 308 (ex 1-10)
- Sách Advanced Engineering Mathematics 10th Edition, Trang 318 (ex 11-22)
- Trang 318 (ex 21-35)

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 261 (ex 8-16) Trang 271 (ex 11-20)

- Peter_V__O'Neil-Advanced_Engineering_Mathematics,_7th_Edition__-Cengage(2011), Trang 261 (ex 8-16)

Linear System, Coefficient Matrix, Augmented Matrix

A **linear system of *m* equations in *n* unknowns** is a set of equations of the form

$$
\begin{aligned}
a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + \cdots + a_{2n}x_n &= b_2 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m.
\end{aligned}
$$

Matrix Form of the Linear System

$$\mathbf{Ax} = \mathbf{b}$$

coefficient matrix

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_n \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

augmented matrix

$$\tilde{\mathbf{A}} = \left[\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & b_1 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right]$$

Gauss Elimination

$$2x_1 + 5x_2 = 2$$

$$-4x_1 + 3x_2 = -30.$$

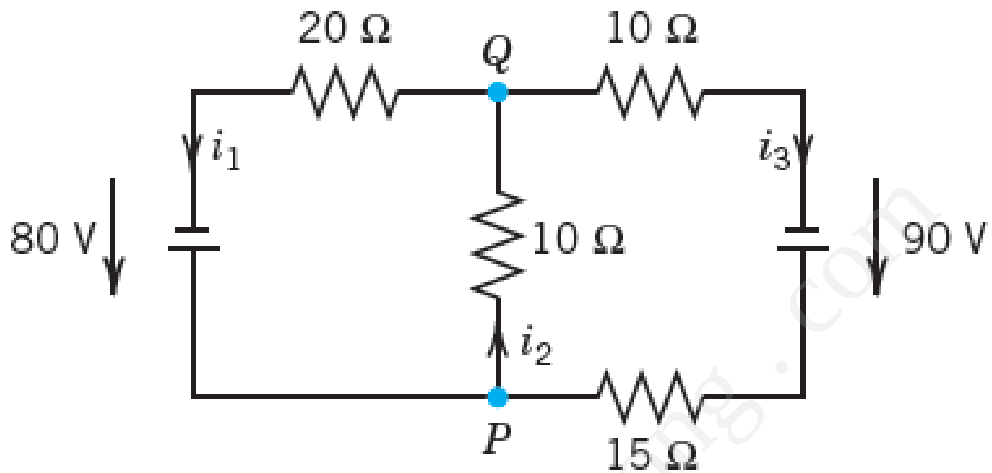
Its augmented matrix is

$$\begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ -4x_1 + 3x_2 &= -30. \end{aligned} \qquad \begin{bmatrix} 2 & 5 & 2 \\ -4 & 3 & -30 \end{bmatrix}$$

Nhân phương trình 1 cho 2 và cộng 2 phương trình

$$\begin{aligned} 2x_1 + 5x_2 &= 2 \\ 13x_2 &= -26 \end{aligned} \qquad \begin{bmatrix} 2 & 5 & 2 \\ 0 & 13 & -26 \end{bmatrix}$$



Node P:	$i_1 -$	$i_2 +$	$i_3 =$	0
Node Q:	$-i_1 +$	$i_2 -$	$i_3 =$	0
Right loop:		$10i_2 +$	$25i_3 =$	90
Left loop:	$20i_1 +$	$10i_2$	$=$	80

Equations

Pivot 1 →

$$\begin{aligned} x_1 - x_2 + x_3 &= 0 \\ -x_1 + x_2 - x_3 &= 0 \\ 10x_2 + 25x_3 &= 90 \\ 20x_1 + 10x_2 &= 80. \end{aligned}$$

Eliminate →

Augmented Matrix $\tilde{\mathbf{A}}$

Pivot 1 →

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Eliminate →

Equations

Pivot 1 →

$$x_1 - x_2 + x_3 = 0$$
$$-x_1 + x_2 - x_3 = 0$$

Eliminate →

$$10x_2 + 25x_3 = 90$$
$$20x_1 + 10x_2 = 80.$$

Augmented Matrix $\tilde{\mathbf{A}}$

Pivot 1 →

Eliminate →

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & 10 & 25 & 90 \\ 20 & 10 & 0 & 80 \end{array}\right]$$

Row 2 + Row 1 Row 4 − 20 Row 1

$$x_1 - x_2 + x_3 = 0$$
$$0 = 0$$
$$10x_2 + 25x_3 = 90$$
$$30x_2 - 20x_3 = 80.$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 10 & 25 & 90 \\ 0 & 30 & -20 & 80 \end{array}\right]$$

$$
\begin{aligned}
x_1 - \quad x_2 + \quad x_3 &= \quad 0 \\
\text{Pivot 10} \longrightarrow \boxed{10x_2} + 25x_3 &= 90 \\
\text{Eliminate } 30x_2 \longrightarrow \boxed{30x_2} - 20x_3 &= 80 \\
0 &= \quad 0.
\end{aligned}
$$

$$
\begin{matrix} \\ \text{Pivot 10} \longrightarrow \\ \text{Eliminate 30} \longrightarrow \\ \\ \end{matrix}
\left[\begin{array}{rrr|r}
1 & -1 & 1 & 0 \\
0 & \boxed{10} & 25 & 90 \\
0 & \boxed{30} & -20 & 80 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Row 3 − 3 Row 2

$$
\begin{aligned}
x_1 - \quad x_2 + \quad x_3 &= \quad 0 \\
10x_2 + 25x_3 &= \quad 90 \\
- \; 95x_3 &= -190 \\
0 &= \quad 0.
\end{aligned}
\qquad
\left[\begin{array}{rrr|r}
1 & -1 & 1 & 0 \\
0 & 10 & 25 & 90 \\
0 & 0 & -95 & -190 \\
0 & 0 & 0 & 0
\end{array}\right]
$$

Determination of (in this order)

$$- 95x_3 = -190$$

$$10x_2 + 25x_3 = 90$$

$$x_1 - x_2 + x_3 = 0$$

$$x_3 = i_3 = 2\,[\mathrm{A}]$$

$$x_2 = \frac{1}{10}(90 - 25x_3) = i_2 = 4\,[\mathrm{A}]$$

$$x_1 = x_2 - x_3 = i_1 = 2\,[\mathrm{A}]$$

- Sinh viên tự đọc thêm ví dụ trang 278-280 Advanced Engineering Mathematics 10th Edition

Linear Independence and Dependence of Vectors

Given any set of *m* vectors $\mathbf{a}_{(1)}, \cdots, \mathbf{a}_{(m)}$:h the same number of components), a **linear combination** of these vectors is an expression of the form

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)}$$

$c_1, c_2, \cdots, c_m$ where are any scalars.

consider the equation

$$c_1\mathbf{a}_{(1)} + c_2\mathbf{a}_{(2)} + \cdots + c_m\mathbf{a}_{(m)} = \mathbf{0}.$$

Example 1

linearly dependent

$$\mathbf{a}_{(1)} = [\;3 \quad 0 \quad 2 \quad 2]$$

$$\mathbf{a}_{(2)} = [-6 \quad 42 \quad 24 \quad 54]$$

$$\mathbf{a}_{(3)} = [\;21 \quad -21 \quad 0 \quad -15]$$

$$6\mathbf{a}_{(1)} - \tfrac{1}{2}\mathbf{a}_{(2)} - \mathbf{a}_{(3)} = \mathbf{0}$$

Rank of a Matrix

The **rank** of a matrix **A** is the maximum number of linearly independent row vectors of **A**

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

has rank 2, because the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix}$$

has rank 2, because Example 1 shows that the first two row vectors are linearly independent, whereas all three row vectors are linearly dependent.

Row-equivalent matrices have the same rank.

Determination of Rank

For the matrix in Example 2 we obtain successively

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 2 & 2 \\ -6 & 42 & 24 & 54 \\ 21 & -21 & 0 & -15 \end{bmatrix} \quad \text{(given)}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & -21 & -14 & -29 \end{bmatrix} \quad \begin{matrix} \\ \text{Row 2} + 2 \text{ Row 1} \\ \text{Row 3} - 7 \text{ Row 1} \end{matrix}$$

$$\begin{bmatrix} 3 & 0 & 2 & 2 \\ 0 & 42 & 28 & 58 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} \\ \\ \text{Row 3} + \tfrac{1}{2} \text{Row 2.} \end{matrix}$$

- Tự học: Sinh viên đọc sách tìm hiểu theorem 2-5 sách Advanced Engineering Mathematics 10th Edition pages 284-285

- Bài tập:Sách Advanced Engineering Mathematics 10th Edition, Trang 287 (ex 1-9) (ex 17-26)

Vector Space R^n

The vector space R^n consisting of all vectors with n components (n real numbers) has dimension n.

Nonhomogeneous Linear System

$$a_{11}x_1 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = b_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{m1}x_1 + \cdots + a_{mn}x_n = b_m.$$

Homogeneous Linear System

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0
\end{aligned}
$$

Second- and Third-Order Determinants

A **determinant of second order** is denoted and defined by

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Determinant

The **determinant** of a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is

defined as (*ad*–*bc*)

$$\begin{vmatrix} 3 & -2 \\ 7 & 4 \end{vmatrix} = (3 \times 4) - (-2 \times 7)$$

$$= 12 - (-14) = \mathbf{26}$$

Determinant (Định thức)

The **determinant** of a 2 by 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as ($ad-bc$)

Cramer's rule for solving linear systems of two equations in two unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1a_{22} - a_{12}b_2}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11}b_2 - b_1a_{21}}{D}$$

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

$$x_1 = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{84}{14} = 6, \quad x_2 = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{-56}{14} = -4$$

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

Third-Order Determinants

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

Cramer's Rule for Linear Systems of Three Equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} \qquad D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Determinants

$$D = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

For $n = 1$, this determinant is defined by

$$D = a_{11}.$$

For $n \geqq 2$ by

$$D = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn} \qquad (j = 1, 2, \cdots, \text{or } n)$$

or

$$D = a_{1k}C_{1k} + a_{2k}C_{2k} + \cdots + a_{nk}C_{nk} \quad (k = 1, 2, \cdots, \text{or } n)$$

Here,

$$C_{jk} = (-1)^{j+k}M_{jk}$$

M_{jk} is a determinant of order $n - 1$

$$D = \sum_{k=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (j = 1, 2, \cdots, \text{or } n)$$

$$D = \sum_{j=1}^{n} (-1)^{j+k} a_{jk} M_{jk} \qquad (k = 1, 2, \cdots, \text{or } n)$$

$$D = \begin{vmatrix} 1 & 3 & 0 \\ 2 & 6 & 4 \\ -1 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 6 & 4 \\ 0 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ -1 & 0 \end{vmatrix}$$

$$= 1(12 - 0) - 3(4 + 4) + 0(0 + 6) = -12.$$

Cramer's Rule

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

$$
x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \cdots, \quad x_n = \frac{D_n}{D}
$$

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 300 (ex 7-15), (ex 17-19), (ex 21-25)

Inverse of a Matrix

The **inverse** of an $n \times n$ matrix $\mathbf{A} = [a_{jk}]$ is denoted by $\mathbf{A}^{-1}$

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ *matrix* $\mathbf{A} = [a_{jk}]$ *is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

Inverse of a Matrix by Determinants

The inverse of a nonsingular $n \times n$ *matrix* $\mathbf{A} = [a_{jk}]$ *is given by*

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}}[C_{jk}]^{\mathsf{T}} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \cdot & \cdot & \cdots & \cdot \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \textit{is} \quad \mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

The inverse or reciprocal of a 3 by 3 matrix

The **adjoint** of a matrix A is obtained by:

(i) forming a matrix B of the cofactors of A, and

(ii) **transposing** matrix B to give B^T, where B^T is the matrix obtained by writing the rows of B as the columns of B^T. Then **adj** $\mathbf{A} = \mathbf{B}^{\mathbf{T}}$

The **inverse of matrix** A, A^{-1} is given by

$$A^{-1} = \frac{\text{adj } A}{|A|}$$

where adj A is the adjoint of matrix A and $|A|$ is the determinant of matrix A.

The inverse or reciprocal of a 3 by 3 matrix

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where adj A is the adjoint of matrix A and $|A|$ is the determinant of matrix A.

Problem 17. Determine the inverse of the matrix

$$\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$$

The inverse of matrix A, $A^{-1} = \frac{\text{adj } A}{|A|}$

The adjoint of A is found by:

(i) obtaining the matrix of the cofactors of the elements, and

(ii) transposing this matrix.

The cofactor of element 3 is $+\begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} = 21$

The cofactor of element 4 is $-\begin{vmatrix} 2 & 7 \\ 1 & -2 \end{vmatrix} = 11$, and so on.

The matrix of cofactors is $\begin{pmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{pmatrix}$

The inverse or reciprocal of a 3 by 3 matrix

Problem 17. Determine the inverse of the matrix

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The matrix of cofactors is $\begin{pmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{pmatrix}$

The transpose of the matrix of cofactors, i.e. the adjoint of the matrix, is obtained by writing the rows as columns, and is $\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$

From Problem 14, the determinant of $\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$ is 113

Hence the inverse of $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}}{113} \text{ or } \frac{1}{113}\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$$

The inverse or reciprocal of a 3 by 3 matrix

Problem 17. Determine the inverse of the matrix

$$\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$$

The inverse of matrix A, $A^{-1} = \dfrac{\text{adj } A}{|A|}$

The adjoint of A is found by:

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The cofactor of element 3 is $+\begin{vmatrix} 0 & 7 \\ -3 & -2 \end{vmatrix} = 21$

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The matrix of cofactors is $\begin{pmatrix} 21 & 11 & -6 \\ 11 & -5 & 13 \\ 28 & -23 & -8 \end{pmatrix}$

The transpose of the matrix of cofactors, i.e. the adjoint of the matrix, is obtained by writing the rows as columns, and is $\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}$

From Problem 14, the determinant of $\begin{vmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{vmatrix}$ is 113

Hence the inverse of $\begin{pmatrix} 3 & 4 & -1 \\ 2 & 0 & 7 \\ 1 & -3 & -2 \end{pmatrix}$ is

$$\frac{\begin{pmatrix} 21 & 11 & 28 \\ 11 & -5 & -23 \\ -6 & 13 & -8 \end{pmatrix}}{113} \text{ or } \frac{1}{113}\begin{pmatrix} \mathbf{21} & \mathbf{11} & \mathbf{28} \\ \mathbf{11} & \mathbf{-5} & \mathbf{-23} \\ \mathbf{-6} & \mathbf{13} & \mathbf{-8} \end{pmatrix}$$

The inverse or reciprocal of a 3 by 3 matrix

Problem 18. Find the inverse of

$$\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$$

$$\text{Inverse} = \frac{\text{adjoint}}{\text{determinant}}$$

The matrix of cofactors is $\begin{pmatrix} -17 & 9 & 15 \\ 23 & -13 & -21 \\ 18 & -10 & -16 \end{pmatrix}$

The transpose of the matrix of cofactors (i.e. the adjoint) is $\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}$

The determinant of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= 1(7-24) - 5(-21+12) - 2(18-3)$$

$$= -17 + 45 - 30 = -2$$

Hence the inverse of $\begin{pmatrix} 1 & 5 & -2 \\ 3 & -1 & 4 \\ -3 & 6 & -7 \end{pmatrix}$

$$= \frac{\begin{pmatrix} -17 & 23 & 18 \\ 9 & -13 & -10 \\ 15 & -21 & -16 \end{pmatrix}}{-2}$$

$$= \begin{pmatrix} \mathbf{8.5} & \mathbf{-11.5} & \mathbf{-9} \\ \mathbf{-4.5} & \mathbf{6.5} & \mathbf{5} \\ \mathbf{-7.5} & \mathbf{10.5} & \mathbf{8} \end{pmatrix}$$

2. Write down the transpose of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

3. Determine the adjoint of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

4. Determine the adjoint of

$$\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$$

5. Find the inverse of

$$\begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$$

6. Find the inverse of $\begin{pmatrix} 3 & 6 & \frac{1}{2} \\ 5 & -\frac{2}{3} & 7 \\ -1 & 0 & \frac{3}{5} \end{pmatrix}$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

$$C_{11} = \begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} = -7, \quad C_{21} = -\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 2, \quad C_{31} = \begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = 3,$$

$$C_{12} = -\begin{vmatrix} 3 & 1 \\ -1 & 4 \end{vmatrix} = -13, \quad C_{22} = \begin{vmatrix} -1 & 2 \\ -1 & 4 \end{vmatrix} = -2, \quad C_{32} = -\begin{vmatrix} -1 & 2 \\ 3 & 1 \end{vmatrix} = 7,$$

$$C_{13} = \begin{vmatrix} 3 & -1 \\ -1 & 3 \end{vmatrix} = 8, \quad C_{23} = -\begin{vmatrix} -1 & 1 \\ -1 & 3 \end{vmatrix} = 2, \quad C_{33} = \begin{vmatrix} -1 & 1 \\ 3 & -1 \end{vmatrix} = -2$$

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}.$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -0.7 & 0.2 & 0.3 \\ -1.3 & -0.2 & 0.7 \\ 0.8 & 0.2 & -0.2 \end{bmatrix}.$$

Cramer's rule for solving linear systems of two equations in two unknowns

$$a_{11}x_1 + a_{12}x_2 = b_1$$

$$a_{21}x_1 + a_{22}x_2 = b_2$$

$$x_1 = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{D} = \frac{b_1 a_{22} - a_{12} b_2}{D}$$

$$x_2 = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{D} = \frac{a_{11} b_2 - b_1 a_{21}}{D}$$

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

$$x_1 = \frac{\begin{vmatrix} 12 & 3 \\ -8 & 5 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{84}{14} = 6$$

$$x_2 = \frac{\begin{vmatrix} 4 & 12 \\ 2 & -8 \end{vmatrix}}{\begin{vmatrix} 4 & 3 \\ 2 & 5 \end{vmatrix}} = \frac{-56}{14} = -4$$

Inverse of matrix for solving linear systems of two equations in two unknowns

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

$$\mathbf{A}\mathbf{A}^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{I}\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$$

$$\mathbf{A} = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 12 \\ -8 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

Inverse of matrix for solving linear systems of two equations in two unknowns

$$4x_1 + 3x_2 = 12$$

$$2x_1 + 5x_2 = -8$$

$$Ax = b$$

$$AA^{-1}x = A^{-1}b$$

$$Ix = A^{-1}b$$

$$x = A^{-1}b$$

$$|A| = |\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}| = 14$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix}5 & -3\\-2 & 4\end{bmatrix}$$

$$|A| = |\begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}| = 14$$

$$A = \begin{bmatrix}4 & 3\\2 & 5\end{bmatrix}$$

$$b = \begin{bmatrix}12\\-8\end{bmatrix}$$

$$x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$x = \begin{bmatrix}x_1\\x_2\end{bmatrix} = A^{-1}b = \frac{1}{14}\begin{bmatrix}5 & -3\\-2 & 4\end{bmatrix}\begin{bmatrix}12\\-8\end{bmatrix} = \begin{bmatrix}6\\-4\end{bmatrix}$$

Inverse of a Matrix by Determinants

$$(\mathbf{AC})^{-1} = \mathbf{C}^{-1}\mathbf{A}^{-1}$$

Cancellation Laws

Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ *be* $n \times n$ *matrices. Then:*

(a) *If* $\text{rank}\ \mathbf{A} = n$ *and* $\mathbf{AB} = \mathbf{AC}$, *then* $\mathbf{B} = \mathbf{C}$.

(b) *If* $\text{rank}\ \mathbf{A} = n$, *then* $\mathbf{AB} = \mathbf{0}$ *implies* $\mathbf{B} = \mathbf{0}$. *Hence if* $\mathbf{AB} = \mathbf{0}$, *but* $\mathbf{A} \neq \mathbf{0}$ *as well as* $\mathbf{B} \neq \mathbf{0}$, *then* $\text{rank}\ \mathbf{A} < n$ *and* $\text{rank}\ \mathbf{B} < n$.

(c) *If* $\mathbf{A}$ *is singular, so are* $\mathbf{BA}$ *and* $\mathbf{AB}$.

Inverse of a Matrix by Determinants

Determinant of a Product of Matrices

For any $n \times n$ *matrices* $\mathbf{A}$ *and* $\mathbf{B}$,

$$\det(\mathbf{AB}) = \det(\mathbf{BA}) = \det \mathbf{A} \det \mathbf{B}.$$

Inner Product Spaces

This product is called the **inner product** or **dot product** of **a** and **b**. Other notations for it are (**a**, **b**) and **a • b**

$$\mathbf{a}^{\mathsf{T}}\mathbf{b} = (\mathbf{a}, \mathbf{b}) = \mathbf{a} \bullet \mathbf{b} = [a_1 \cdots a_n] \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \sum_{i=1}^{n} a_l b_l = a_1 b_1 + \cdots + a_n b_n$$

Inner Product Spaces

The *length* or **norm** of a vector in V is defined by

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} \quad (\geqq 0)$$

A vector of norm 1 is called a **unit vector**.

Inner Product Spaces

$$|(\mathbf{a}, \mathbf{b})| \leqq \|\mathbf{a}\| \, \|\mathbf{b}\| \qquad (\textit{Cauchy–Schwarz}^5 \textit{ inequality}).$$

$$\|\mathbf{a} + \mathbf{b}\| \leqq \|\mathbf{a}\| + \|\mathbf{b}\| \qquad (\textit{Triangle inequality}).$$

$$\|\mathbf{a} + \mathbf{b}\|^2 + \|\mathbf{a} - \mathbf{b}\|^2 = 2(\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2) \qquad (\textit{Parallelogram equality}).$$

Inner Product Spaces

$$(\mathbf{a}, \mathbf{b}) = \mathbf{a}^{\mathsf{T}}\mathbf{b} = a_1 b_1 + \cdots + a_n b_n$$

$$\|\mathbf{a}\| = \sqrt{(\mathbf{a}, \mathbf{a})} = \sqrt{\mathbf{a}^{\mathsf{T}}\mathbf{a}} = \sqrt{a_1^2 + \cdots + a_n^2}$$

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 308 (ex 1-10)
- Sách Advanced Engineering Mathematics 10th Edition, Trang 318 (ex 11-28) Trang 318 (ex 21-35)

Linear Algebra:
Matrix Eigenvalue Problems

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

A is a given square matrix, an unknown scalar, and **x** an unknown vector.

Determination of Eigenvalues and Eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$a_{11}x_1 + \cdots + a_{1n}x_n = \lambda x_1$$

$$a_{21}x_1 + \cdots + a_{2n}x_n = \lambda x_2$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{n1}x_1 + \cdots + a_{nn}x_n = \lambda x_n$$

Determination of Eigenvalues and Eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{aligned}
(a_{11} - \lambda)x_1 + \quad a_{12}x_2 \quad + \cdots + \quad a_{1n}x_n \quad &= 0 \\
a_{21}x_1 \quad + (a_{22} - \lambda)x_2 + \cdots + \quad a_{2n}x_n \quad &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\
a_{n1}x_1 \quad + \quad a_{n2}x_2 \quad + \cdots + (a_{nn} - \lambda)x_n &= 0.
\end{aligned}$$

Determination of Eigenvalues and Eigenvectors

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$$

$$\begin{aligned}
(a_{11} - \lambda)x_1 + \quad a_{12}x_2 \quad + \cdots + \quad a_{1n}x_n \quad &= 0 \\
a_{21}x_1 \quad + (a_{22} - \lambda)x_2 + \cdots + \quad a_{2n}x_n \quad &= 0 \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}x_1 \quad + \quad a_{n2}x_2 \quad + \cdots + (a_{nn} - \lambda)x_n &= 0.
\end{aligned}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

Determination of Eigenvalues and Eigenvectors

characteristic equation

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}.$$

characteristic matrix

$$D(\lambda) = \det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0$$

characteristic determinant

Determination of Eigenvalues and Eigenvectors

characteristic equation

$$(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = \mathbf{0}.$$

The eigenvalues of a square matrix **A** *are the roots of the characteristic equation*

x *are eigenvectors of a matrix* **A**

Eigenvalues and eigenvectors

Problem 9. Determine the eigenvalues of the matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

The eigenvalue is determined by solving the characteristic equation $|A - \boldsymbol{\lambda} I| = \mathbf{0}$

i.e. $$\left| \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

i.e. $$\left| \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = 0$$

i.e. $$\begin{vmatrix} 3-\lambda & 4 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

Hence, $(3-\lambda)(1-\lambda) - (2)(4) = 0$

i.e. $3 - 3\lambda - \lambda + \lambda^2 - 8 = 0$

and $\lambda^2 - 4\lambda - 5 = 0$

i.e. $(\lambda - 5)(\lambda + 1) = 0$

from which, $\lambda - 5 = 0$ i.e. $\boldsymbol{\lambda} = \mathbf{5}$ or $\lambda + 1 = 0$ i.e. $\boldsymbol{\lambda} = -\mathbf{1}$

(Instead of factorising, the quadratic formula could be used; even electronic calculators can solve quadratic equations.)

Hence, the eigenvalues of the matrix $\begin{pmatrix} \mathbf{3} & \mathbf{4} \\ \mathbf{2} & \mathbf{1} \end{pmatrix}$ **are 5 and –1**

Eigenvalues and eigenvectors

Problem 10. Determine the eigenvectors of the matrix $A = \begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$

From Problem 9, the eigenvalues of $\begin{pmatrix} 3 & 4 \\ 2 & 1 \end{pmatrix}$ are $\lambda_1 = 5$ and $\lambda_2 = -1$

Using the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda_1 = 5$

then $\begin{pmatrix} 3-5 & 4 \\ 2 & 1-5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e.

$$\begin{pmatrix} -2 & 4 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Using the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for $\lambda_2 = -1$

then $\begin{pmatrix} 3 - -1 & 4 \\ 2 & 1 - -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

i.e.

$$\begin{pmatrix} 4 & 4 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

from which,

$$4x_1 + 4x_2 = 0$$

and

$$2x_1 + 2x_2 = 0$$

From either of these two equations, $x_1 = -x_2$ or $x_2 = -x_1$

Hence, whatever value x_1 is, the value of x_2 will be –1 times greater. Hence the simplest eigenvector is: $x_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Summarising, $\mathbf{x}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to $\boldsymbol{\lambda}_1 = \mathbf{5}$ **and** $\mathbf{x}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to $\boldsymbol{\lambda}_2 = -\mathbf{1}$

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$$

$$-\lambda^3 - \lambda^2 + 21\lambda + 45 = 0$$

$$\lambda_1 = 5, \lambda_2 = \lambda_3 = -3$$

Find the eigenvalues and eigenvectors of

$$\mathbf{A} = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix} \qquad \lambda_1 = 5, \lambda_2 = \lambda_3 = -3$$

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} - 5\mathbf{I} = \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}$$

It row-reduces to

$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

Find the eigenvalues and eigenvectors of

It row-reduces to
$$\begin{bmatrix} -7 & 2 & -3 \\ 0 & -\frac{24}{7} & -\frac{48}{7} \\ 0 & 0 & 0 \end{bmatrix}$$

Choosing $x_3 = -1$ we have $x_2 = 2$ from $-\frac{24}{7}x_2 - \frac{48}{7}x_3 = 0$ and then $x_1 = 1$

$\lambda = 5$ is $\mathbf{x}_1 = [1 \quad 2 \quad -1]^T$

Find the eigenvalues and eigenvectors of

$$\mathbf{A} - \lambda\mathbf{I} = \mathbf{A} + 3\mathbf{I} = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}$$

row-reduces to $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Find the eigenvalues and eigenvectors of

row-reduces to $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

From $x_1 + 2x_2 - 3x_3 = 0$ we have $x_1 = -2x_2 + 3x_3$.

Choosing $x_2 = 1, x_3 = 0$ and $x_2 = 0, x_3 = 1$

Eigenvalues and eigenvectors

4. $\begin{pmatrix} -1 & -1 & 1 \\ -4 & 2 & 4 \\ -1 & 1 & 5 \end{pmatrix}$ 5. $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

6. $\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ 7. $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \\ -1 & 1 & 3 \end{pmatrix}$

- Tự học: 8.2 Some Applications of Eigenvalue Problems, Sách Advanced Engineering Mathematics 10th Edition, Trang 329 (ex 1-16)

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 329 (ex 1-16)

Symmetric, Skew-Symmetric, and Orthogonal Matrices

A ***real*** square matrix $\mathbf{A} = [a_{jk}]$ is called

symmetric if transposition leaves it unchanged,

(1) $$\mathbf{A}^\mathsf{T} = \mathbf{A}, \qquad \text{thus} \qquad a_{kj} = a_{jk},$$

skew-symmetric if transposition gives the negative of $\mathbf{A}$,

(2) $$\mathbf{A}^\mathsf{T} = -\mathbf{A}, \qquad \text{thus} \qquad a_{kj} = -a_{jk},$$

orthogonal if transposition gives the inverse of $\mathbf{A}$,

(3) $$\mathbf{A}^\mathsf{T} = \mathbf{A}^{-1}.$$

Symmetric, Skew-Symmetric, and Orthogonal Matrices

$$\begin{bmatrix} -3 & 1 & 5 \\ 1 & 0 & -2 \\ 5 & -2 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 9 & -12 \\ -9 & 0 & 20 \\ 12 & -20 & 0 \end{bmatrix}, \quad \begin{bmatrix} \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} & -\frac{2}{3} \end{bmatrix}$$

Quadratic Forms

$$Q = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = \sum_{j=1}^{n}\sum_{k=1}^{n} a_{jk}x_j x_k$$

$$\begin{aligned}
= \quad & a_{11}x_1^2 + a_{12}x_1x_2 + \cdots + a_{1n}x_1x_n \\
& + a_{21}x_2x_1 + a_{22}x_2^2 + \cdots + a_{2n}x_2x_n \\
& + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
& + a_{n1}x_nx_1 + a_{n2}x_nx_2 + \cdots + a_{nn}x_n^2
\end{aligned}$$

Quadratic Forms

$$\mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x} = [x_1 \quad x_2]\begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2$$

$$3x_1^2 + 4x_1x_2 + 6x_2x_1 + 2x_2^2 = 3x_1^2 + 10x_1x_2 + 2x_2^2.$$

Bài tập

- Sách Advanced Engineering Mathematics 10th Edition, Trang 338 (ex 1-10)
- Sách Advanced Engineering Mathematics 10th Edition, Trang 345 (ex 1-5), (ex 9-16)