UNIVERSITY OF TECHNOLOGY AND EDUCATION
FACULTY OF MECHANICAL ENGINEERING
DEPARTMENT OF MECHATRONICS

Approximation Methods for Solving Differential Equations

AMME_131529

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f(t) là hàm số f theo biến t

$$f(t) = 4t + 2$$

f(t,y) làm hàm số f bao gồm 2 biến t và y

$$f(t, y) = 4t + 3t^2 + 2y$$

Cho phương trình vi phân bậc nhất (First order diferential equation)

$$\frac{dy}{dt} = y' = f(t, y) = 2t + 3y$$

$$1 \le t \le 5 \qquad y(1) = 2$$

Find y(5) có nghĩa là tìm giá trị hàm y(t) tại t=5

Approximation Methods for Solving Differential Equations

- **Euler's Method**

The object of Euler's method is to obtain approximations to the well-posed initial-value problem

$$\frac{dy}{dt} = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha$$

y dependent variable (biến phụ thuộc)
t independent variable (biến độc lập)

Cho phương trình vi phân bậc nhất (First order diferential equation)

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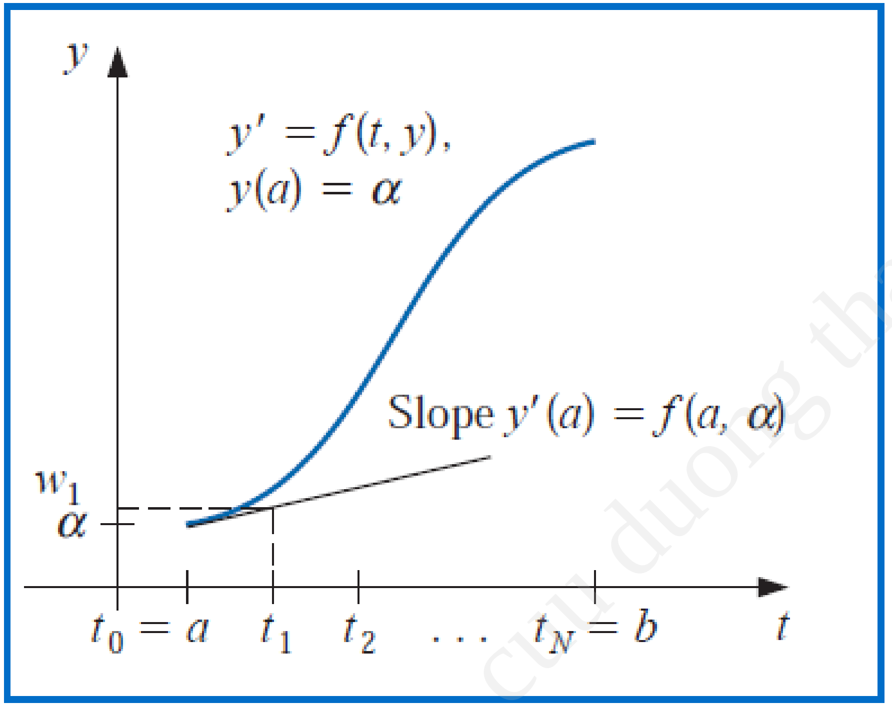
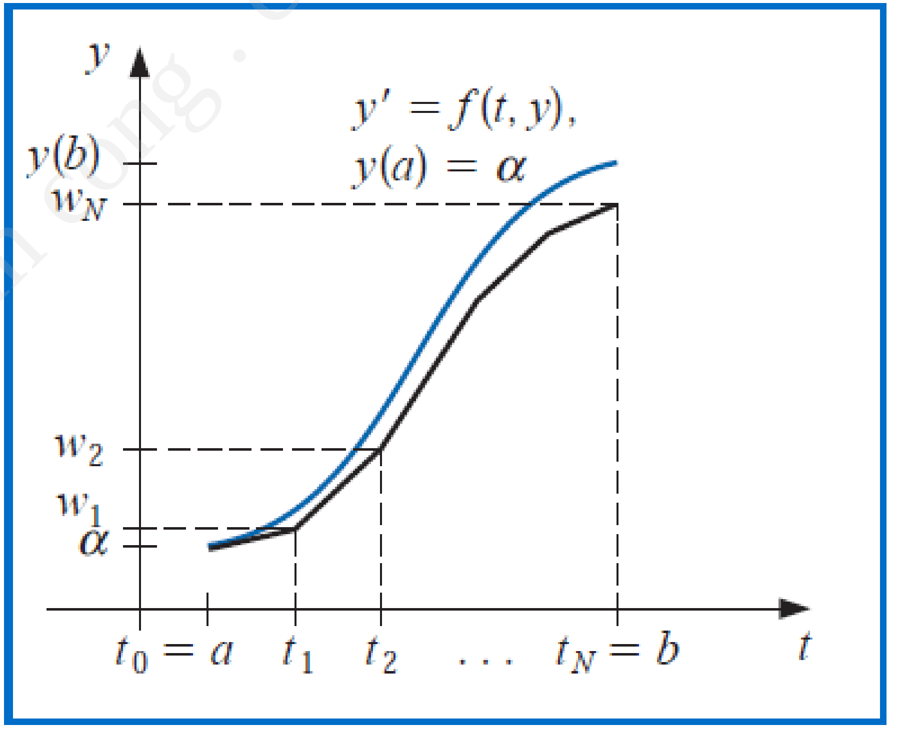
$$1 \le t \le 5 \qquad y(1) = 2$$

Find y(5) có nghĩa là tìm giá trị hàm y(t) tại t=5

One step in Euler's method and a series of steps



One step



A series of steps

A continuous approximation to the solution *y(t)* will not be obtained;

Approximations to *y* will be generated at various values, called **mesh points**, in the interval [*a*, *b*].

Choosing a positive integer *N* and selecting the mesh points

$$t_i = a + ih, \quad \text{for each } i = 0, 1, 2, \ldots, N$$

The common distance between the points $h = (b - a)/N = t_{i+1} - t_i$

is called the **step size**

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We will use Taylor's Theorem to derive Euler's method

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)y'(t_i) + \frac{(t_{i+1} - t_i)^2}{2}y''(\xi_i),$$

for some number ξ_i in (t_i, t_{i+1}). Because $h = t_{i+1} - t_i$, we have

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(\xi_i),$$

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$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, y'(t_i)$$

$$y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i)\, y'(t_i)$$

$$\Leftrightarrow y(t_{i+1}) = y(t_i) + h y'(t_i)$$

Euler's method constructs $w_i \approx y(t_i)$ for each $i = 1, 2, \ldots,$ N, by deleting the remainder term. Thus Euler's method is

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + h f(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N - 1.$$

Example: we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

$$h = 0.5 \Rightarrow N = \frac{b-a}{h} = \frac{2-0}{0.5} = 4$$

Find y(2) (Tìm giá trị của y(t) khi t=2)

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at $t = 2$

$$t_0 = 0 + 0x0.5 = 0 \qquad y(t = t_0) = y(0) \approx w_0$$

$$t_1 = 0 + 1x0.5 = 0.5 \qquad y(t = t_1) = y(0.5) \approx w_1$$

$$t_2 = 0 + 2x0.5 = 1 \qquad y(t = t_2) = y(1) \approx w_2$$

$$t_3 = 0 + 3x0.5 = 1.5 \qquad y(t = t_3) = y(1.5) \approx w_3$$

$$t_4 = 0 + 2x0.5 = 2 \qquad y(t = t_4) = y(2) \approx w_4$$

Example: we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5 \qquad h = 0.5 \Rightarrow N = \frac{b-a}{h} = \frac{2-0}{0.5} = 4$$

Find y(2) (Tìm giá trị của y(t) khi t=2)

For this problem $f(t, y) = y - t^2 + 1$, so

$$w_0 = y(0) = 0.5;$$

$$w_1 = w_0 + 0.5\left(w_0 - (0.0)^2 + 1\right) = 0.5 + 0.5(1.5) = 1.25;$$

$$w_2 = w_1 + 0.5\left(w_1 - (0.5)^2 + 1\right) = 1.25 + 0.5(2.0) = 2.25;$$

$$w_3 = w_2 + 0.5\left(w_2 - (1.0)^2 + 1\right) = 2.25 + 0.5(2.25) = 3.375;$$

and

$$y(2) \approx w_4 = w_3 + 0.5\left(w_3 - (1.5)^2 + 1\right) = 3.375 + 0.5(2.125) = 4.4375.$$

Example:Use Euler's method to numerically integrate equation.

$$\frac{dy}{dx} = -2x^3 + 12x^2 - 20x + 8.5$$

from x = 0 to x = 4 with a step size of 0.5. The initial condition at x = 0 is y = 1. Recall

$$y = -0.5x^4 + 4x^3 - 10x^2 + 8.5x + 1$$

B1: Dùng biểu thức 1 và phương pháp Euler để tìm y tại t=4 (đây là kết quả gần đúng)

B2: Biểu thức thứ 2 là nghiệm của phương trình vi phân (biểu thức thứ 1). Tìm kết quả chính xác của y tại x= 4 bằng cách thay x=4 vào biểu thức thức thứ 2.

B3: So sánh kết quả thu được từ B1 và B2

Solution.

$$y(0.5) = y(0) + f(0, 1)0.5$$

where $y(0) = 1$ and the slope estimate at $x = 0$ is

$$f(0, 1) = -2(0)^3 + 12(0)^2 - 20(0) + 8.5 = 8.5$$

Therefore,

$$y(0.5) = 1.0 + 8.5(0.5) = 5.25$$

The true solution at $x = 0.5$ is

$$y = -0.5(0.5)^4 + 4(0.5)^3 - 10(0.5)^2 + 8.5(0.5) + 1 = 3.21875$$

Thus, the error is

$$E_t = \text{true} - \text{approximate} = 3.21875 - 5.25 = -2.03125$$

or, expressed as percent relative error, $\varepsilon_t = -63.1\%$. For the second step,

$$y(1) = y(0.5) + f(0.5, 5.25)0.5$$
$$= 5.25 + [-2(0.5)^3 + 12(0.5)^2 - 20(0.5) + 8.5]0.5$$
$$= 5.875$$

Euler's

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

OUTPUT approximation w to y at the $(N + 1)$ values of t.

Step 1 Set $h = (b - a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

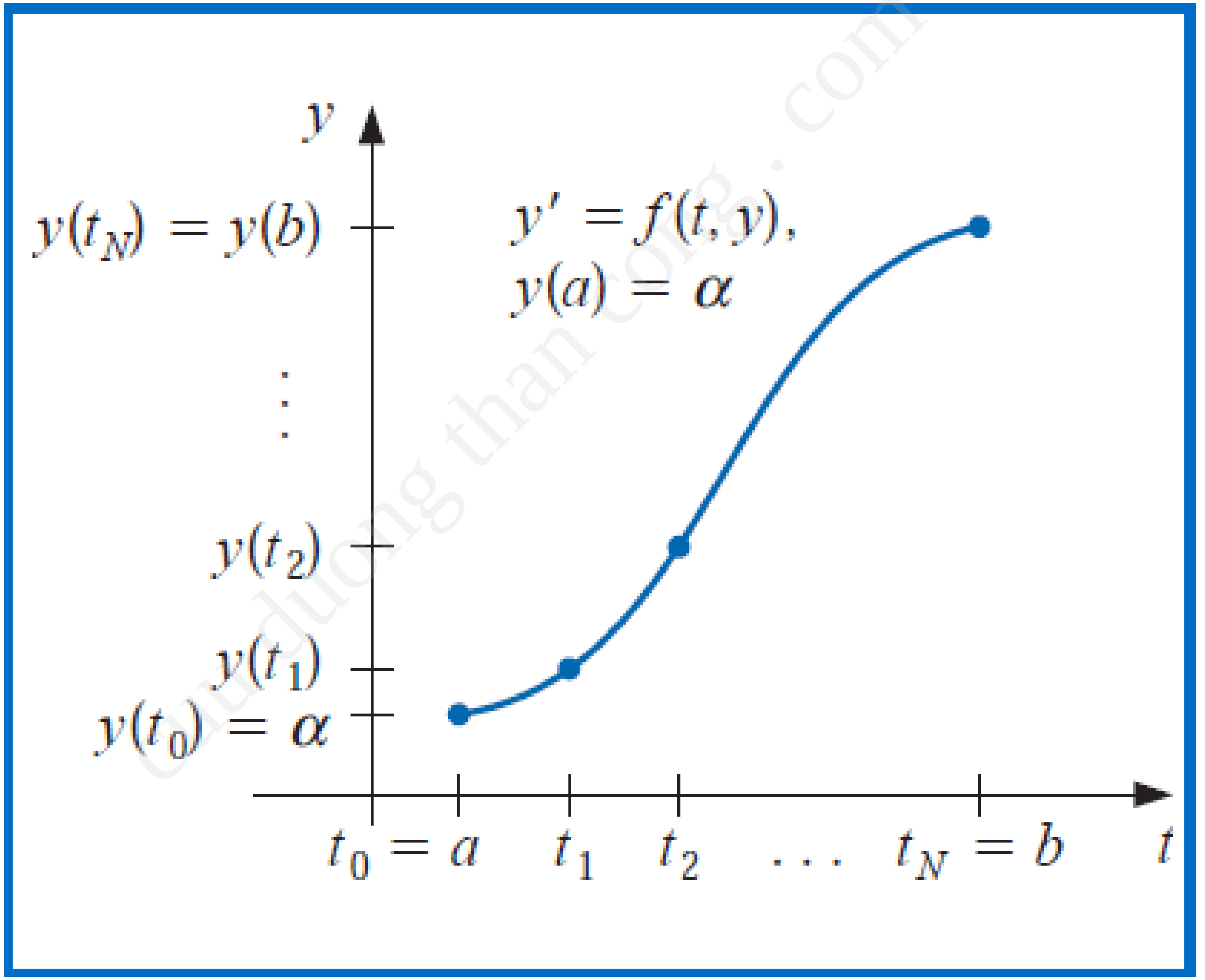
Step 2 For $i = 1, 2, \ldots, N$ do Steps 3, 4.

Step 3 Set $w = w + hf(t, w)$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 4 OUTPUT (t, w).

Step 5 STOP.

The graph of the function highlighting *y(ti)*



Các bạn nên rút gọn biểu thức trước khi thế số sẽ tiết kiệm thời gian và tính toán nhanh hơn

Ví dụ: Biểu thức ban đầu $y(t) = t^2 + 4t - 2 + 6t + 4 - t^2$

Các bạn phải đi tìm y(t=0), y(t=1),y(t=2)……

Các bạn có thể thế lần lượt các giá trị t vào biểu thức trên để tìm y. Bạn sẽ ra kết quả đúng nhưng sẽ tốn nhiều thời gian và dễ mắc sai sót khi tính toán phức tạp.

Các bạn có thể tiết kiệm thời gian và tăng việc tính toán chính xác bằng cách rút gọn biểu thức trước khi tính.

$$y(t) = t^2 + 4t - 2 + 6t + 4 - t^2 = 10t + 2 \Leftrightarrow y(t) = 10t + 2$$

Các bạn có thể thấy biểu thức cuối gọn hơn rất nhiều và khi bạn thế giá trị t vào , các bạn có thể tìm giá trị nhanh hơn và giảm thiểu được việc tính toán sai.

Example: we will use an algorithm for Euler's method to approximate the solution to

$$y' = y - t^2 + 1, \quad 0 \leq t \leq 2, \quad y(0) = 0.5$$

with N = 10 to determine approximations, and compare these with the exact values given by

$$y(t) = (t + 1)^2 - 0.5e^t$$

Solution With $N = 10$ we have $h = 0.2$, $t_i = 0.2i$, $w_0 = 0.5$, and

$$w_{i+1} = w_i + h(w_i - t_i^2 + 1) = w_i + 0.2[w_i - 0.04i^2 + 1] = 1.2w_i - 0.008i^2 + 0.2,$$

for $i = 0, 1, \ldots, 9$. So

$$w_1 = 1.2(0.5) - 0.008(0)^2 + 0.2 = 0.8; \quad w_2 = 1.2(0.8) - 0.008(1)^2 + 0.2 = 1.152;$$

t_i	w_i	$y_i = y(t_i)$	$\lvert y_i - w_i \rvert$
0.0	0.5000000	0.5000000	0.0000000
0.2	0.8000000	0.8292986	0.0292986
0.4	1.1520000	1.2140877	0.0620877
0.6	1.5504000	1.6489406	0.0985406
0.8	1.9884800	2.1272295	0.1387495
1.0	2.4581760	2.6408591	0.1826831
1.2	2.9498112	3.1799415	0.2301303
1.4	3.4517734	3.7324000	0.2806266
1.6	3.9501281	4.2834838	0.3333557
1.8	4.4281538	4.8151763	0.3870225
2.0	4.8657845	5.3054720	0.4396874

Approximation Methods for Solving Differential Equations

- **Higher-Order Taylor Methods**

Euler's method was derived by using Taylor's Theorem with $n = 1$ to approximate the solution of the differential equation.

Find methods for improving the convergence properties of difference methods is to extend this technique of derivation to larger values of n.

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i).$$

Approximation Methods for Solving Differential Equations

- **Higher-Order Taylor Methods**

$$y(t_{i+1}) = y(t_i) + hy'(t_i) + \frac{h^2}{2}y''(t_i) + \cdots + \frac{h^n}{n!}y^{(n)}(t_i) + \frac{h^{n+1}}{(n+1)!}y^{(n+1)}(\xi_i).$$

$$y(t_{i+1}) = y(t_i) + hf(t_i, y(t_i)) + \frac{h^2}{2}f'(t_i, y(t_i)) + \cdots$$
$$+ \frac{h^n}{n!}f^{(n-1)}(t_i, y(t_i)) + \frac{h^{n+1}}{(n+1)!}f^{(n)}(\xi_i, y(\xi_i)).$$

Taylor method of order *n*

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + hT^{(n)}(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

where

$$T^{(n)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \cdots + \frac{h^{n-1}}{n!} f^{(n-1)}(t_i, w_i)$$

Example

Apply Taylor's method of orders

(a) Two

(b) Four

To find y(2) with N = 10 to the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5$$

Solution (a) For the method of order two we need the first derivative of $f(t, y(t)) = y(t) - t^2 + 1$ with respect to the variable t. Because $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = \frac{d}{dt}(y - t^2 + 1) = y' - 2t = y - t^2 + 1 - 2t,$$

so

$$T^{(2)}(t_i, w_i) = f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) = w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i)$$

$$= \left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i$$

Because $N = 10$ we have $h = 0.2$, and $t_i = 0.2i$ for each $i = 1, 2, \ldots, 10$. Thus the second-order method becomes

$$w_0 = 0.5,$$

$$w_{i+1} = w_i + h\left[\left(1 + \frac{h}{2}\right)(w_i - t_i^2 + 1) - ht_i\right]$$

$$= w_i + 0.2\left[\left(1 + \frac{0.2}{2}\right)(w_i - 0.04i^2 + 1) - 0.04i\right]$$

$$= 1.22w_i - 0.0088i^2 - 0.008i + 0.22.$$

The first two stepsgive the approximations

$$y(0.2) \approx w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.22 = 0.83;$$

$$y(0.4) \approx w_2 = 1.22(0.83) - 0.0088(0.2)^2 - 0.008(0.2) + 0.22 = 1.2158$$

t_i	Taylor Order 2 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.500000	0
0.2	0.830000	0.000701
0.4	1.215800	0.001712
0.6	1.652076	0.003135
0.8	2.132333	0.005103
1.0	2.648646	0.007787
1.2	3.191348	0.011407
1.4	3.748645	0.016245
1.6	4.306146	0.022663
1.8	4.846299	0.031122
2.0	5.347684	0.042212

(b) For Taylor's method of order four we need the first three derivatives of $f(t, y(t))$ with respect to t. Again using $y' = y - t^2 + 1$ we have

$$f'(t, y(t)) = y - t^2 + 1 - 2t,$$

$$f''(t, y(t)) = \frac{d}{dt}(y - t^2 + 1 - 2t) = y' - 2t - 2$$

$$= y - t^2 + 1 - 2t - 2 = y - t^2 - 2t - 1,$$

and

$$f'''(t, y(t)) = \frac{d}{dt}(y - t^2 - 2t - 1) = y' - 2t - 2 = y - t^2 - 2t - 1,$$

so

$$\begin{aligned} T^{(4)}(t_i, w_i) &= f(t_i, w_i) + \frac{h}{2} f'(t_i, w_i) + \frac{h^2}{6} f''(t_i, w_i) + \frac{h^3}{24} f'''(t_i, w_i) \\ &= w_i - t_i^2 + 1 + \frac{h}{2}(w_i - t_i^2 + 1 - 2t_i) + \frac{h^2}{6}(w_i - t_i^2 - 2t_i - 1) \\ &\quad + \frac{h^3}{24}(w_i - t_i^2 - 2t_i - 1) \\ &= \left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right)(ht_i) \\ &\quad + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}. \end{aligned}$$

Hence Taylor's method of order four is

$$w_0 = 0.5,$$

$$w_{i+1} = w_i + h\left[\left(1 + \frac{h}{2} + \frac{h^2}{6} + \frac{h^3}{24}\right)(w_i - t_i^2) - \left(1 + \frac{h}{3} + \frac{h^2}{12}\right) h\right.$$

$$\left. + 1 + \frac{h}{2} - \frac{h^2}{6} - \frac{h^3}{24}\right],$$

for $i = 0, 1, \ldots, N - 1$.

Because $N = 10$ and $h = 0.2$ the method becomes

$$w_{i+1} = w_i + 0.2\left[\left(1 + \frac{0.2}{2} + \frac{0.04}{6} + \frac{0.008}{24}\right)(w_i - 0.04i^2)\right.$$

$$\left. - \left(1 + \frac{0.2}{3} + \frac{0.04}{12}\right)(0.04i) + 1 + \frac{0.2}{2} - \frac{0.04}{6} - \frac{0.008}{24}\right]$$

$$= 1.2214w_i - 0.008856i^2 - 0.00856i + 0.2186,$$

for each $i = 0, 1, \ldots, 9$. The first two steps give the approximations

$$y(0.2) \approx w_1 = 1.2214(0.5) - 0.008856(0)^2 - 0.00856(0) + 0.2186 = 0.8293;$$

$$y(0.4) \approx w_2 = 1.2214(0.8293) - 0.008856(0.2)^2 - 0.00856(0.2) + 0.2186 = 1.214091$$

t_i	Taylor Order 4 w_i	Error $\lvert y(t_i) - w_i \rvert$
0.0	0.500000	0
0.2	0.829300	0.000001
0.4	1.214091	0.000003
0.6	1.648947	0.000006
0.8	2.127240	0.000010
1.0	2.640874	0.000015
1.2	3.179964	0.000023
1.4	3.732432	0.000032
1.6	4.283529	0.000045
1.8	4.815238	0.000062
2.0	5.305555	0.000083

• Runge-Kutta Methods

The Taylor methods outlined in the previous section have the desirable property of high-order local truncation error, but the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$.

Runge-Kutta methods have the high-order local truncation error of the Taylor methods but eliminate the need to compute and evaluate the derivatives of $f(t, y)$.

Runge-Kutta Methods of OrderTwo

The first step in deriving a Runge-Kutta method is to determine values for a_1, α_1, and β_1 with the property that $a_1 f(t + \alpha_1, y + \beta_1)$ approximates

$$T^{(2)}(t, y) = f(t, y) + \frac{h}{2} f'(t, y),$$

with error no greater than $O(h^2)$, which is same as the order of the local truncation error for the Taylor method of order two. Since

$$f'(t, y) = \frac{df}{dt}(t, y) = \frac{\partial f}{\partial t}(t, y) + \frac{\partial f}{\partial y}(t, y) \cdot y'(t) \quad \text{and} \quad y'(t) = f(t, y),$$

Khai triển Taylor bậc 2

$$f(t + h/2, y + h/2)$$

$$T^{(2)}(t, y) = hf\left(t + h/2, y + \frac{h}{2} f(t, y)\right)$$

The difference-equation method resulting from replacing $T^{(2)}(t, y)$ in Taylor's method of order two by $f(t + (h/2), y + (h/2) f(t, y))$ is a specific Runge-Kutta method known as the *Midpoint method*.

Midpoint Method (Phương pháp Kutta bậc 2)

$$w_0 = \alpha$$

$$w_{i+1} = w_i + hf\left(t_i + h/2, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

$$\text{for} \quad i = 0, 1, 2, \ldots, N-1$$

Modified Euler Method

$$w_0 = \alpha,$$

(Phương pháp Euler cải tiến)

$$w_{i+1} = w_i + \frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))], \quad \text{for} \quad i = 0, 1, \ldots, N-1$$

Use the Midpoint method and the Modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Use the Midpoint method and the Modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

$$w_0 = \alpha$$

$$w_{i+1} = w_i + hf\left(t_i + h/2, w_i + \frac{h}{2} f(t_i, w_i)\right)$$

$$\text{for} \qquad i = 0, 1, 2, \ldots, N-1$$

Example

Use the Midpoint method and the Modified Euler method with $N = 10$, $h = 0.2$, $t_i = 0.2i$, and $w_0 = 0.5$ to approximate the solution to our usual example,

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution The difference equations produced from the various formulas are

$$\text{Midpoint method:} \quad w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.218;$$

$$\text{Modified Euler method:} \quad w_{i+1} = 1.22w_i - 0.0088i^2 - 0.008i + 0.216,$$

for each $i = 0, 1, \ldots, 9$. The first two steps of these methods give

$$\text{Midpoint method:} \quad w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.218 = 0.828;$$

$$\text{Modified Euler method:} \quad w_1 = 1.22(0.5) - 0.0088(0)^2 - 0.008(0) + 0.216 = 0.826,$$

t_i	$y(t_i)$	Midpoint Method	Error	Modified Euler Method	Error
0.0	0.5000000	0.5000000	0	0.5000000	0
0.2	0.8292986	0.8280000	0.0012986	0.8260000	0.0032986
0.4	1.2140877	1.2113600	0.0027277	1.2069200	0.0071677
0.6	1.6489406	1.6446592	0.0042814	1.6372424	0.0116982
0.8	2.1272295	2.1212842	0.0059453	2.1102357	0.0169938
1.0	2.6408591	2.6331668	0.0076923	2.6176876	0.0231715
1.2	3.1799415	3.1704634	0.0094781	3.1495789	0.0303627
1.4	3.7324000	3.7211654	0.0112346	3.6936862	0.0387138
1.6	4.2834838	4.2706218	0.0128620	4.2350972	0.0483866
1.8	4.8151763	4.8009586	0.0142177	4.7556185	0.0595577
2.0	5.3054720	5.2903695	0.0151025	5.2330546	0.0724173

The Midpoint method is superior to the Modified Euler method in this example

- **Higher-Order Runge-Kutta Methods**

The term $T^{(3)}(t, y)$ can be approximated with error $O(h^3)$ by an expression of the form

$$f(t + \alpha_1, y + \delta_1 f(t + \alpha_2, y + \delta_2 f(t, y))),$$

involving four parameters, the algebra involved in the determination of $\alpha_1, \delta_1, \alpha_2$, and δ_2 is quite involved. The most common $O(h^3)$ is Heun's method, given by

$$w_0 = \alpha$$

$$w_{i+1} = w_i + \frac{h}{4}\left(f(t_i, w_i) + 3f\left(t_i + \frac{2h}{3}, w_i + \frac{2h}{3} f\left(t_i + \frac{h}{3}, w_i + \frac{h}{3} f(t_i, w_i)\right)\right)\right),$$

for $\quad i = 0, 1, \ldots, N-1$

Runge-Kutta Order Four

$$w_0 = \alpha,$$

$$k_1 = hf(t_i, w_i),$$

$$k_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_1\right),$$

$$k_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{1}{2}k_2\right),$$

$$k_4 = hf(t_{i+1}, w_i + k_3),$$

$$w_{i+1} = w_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4),$$

Runge-Kutta (Order Four)

To approximate the solution of the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

at $(N + 1)$ equally spaced numbers in the interval $[a, b]$:

INPUT endpoints a, b; integer N; initial condition α.

Step 1 Set $h = (b - a)/N$;
$t = a$;
$w = \alpha$;
OUTPUT (t, w).

Step 2 For $i = 1, 2, \ldots, N$ do Steps 3–5.

Step 3 Set $K_1 = hf(t, w)$;
$K_2 = hf(t + h/2, w + K_1/2)$;
$K_3 = hf(t + h/2, w + K_2/2)$;
$K_4 = hf(t + h, w + K_3)$.

Step 4 Set $w = w + (K_1 + 2K_2 + 2K_3 + K_4)/6$; (*Compute* w_i.)
$t = a + ih$. (*Compute* t_i.)

Step 5 OUTPUT (t, w).

Step 6 STOP.

Example

Use the Runge-Kutta method of order four with $h = 0.2$, $N = 10$, and $t_i = 0.2i$ to obtain approximations to the solution of the initial-value problem

$$y' = y - t^2 + 1, \quad 0 \le t \le 2, \quad y(0) = 0.5.$$

Solution The approximation to $y(0.2)$ is obtained by

$$w_0 = 0.5$$

$$k_1 = 0.2 f(0, 0.5) = 0.2(1.5) = 0.3$$

$$k_2 = 0.2 f(0.1, 0.65) = 0.328$$

$$k_3 = 0.2 f(0.1, 0.664) = 0.3308$$

$$k_4 = 0.2 f(0.2, 0.8308) = 0.35816$$

$$w_1 = 0.5 + \frac{1}{6}(0.3 + 2(0.328) + 2(0.3308) + 0.35816) = 0.8292933.$$

t_i	Exact $y_i = y(t_i)$	Runge-Kutta Order Four w_i	Error $\|y_i - w_i\|$
0.0	0.5000000	0.5000000	0
0.2	0.8292986	0.8292933	0.0000053
0.4	1.2140877	1.2140762	0.0000114
0.6	1.6489406	1.6489220	0.0000186
0.8	2.1272295	2.1272027	0.0000269
1.0	2.6408591	2.6408227	0.0000364
1.2	3.1799415	3.1798942	0.0000474
1.4	3.7324000	3.7323401	0.0000599
1.6	4.2834838	4.2834095	0.0000743
1.8	4.8151763	4.8150857	0.0000906
2.0	5.3054720	5.3053630	0.0001089

READ Runge-Kutta-Fehlberg method

1. Use the Modified Euler method, Runge-Kutta method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with $h = 0.5$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.

b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with $h = 0.5$; actual solution $y(t) = t + \frac{1}{1-t}$.

c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$; actual solution $y(t) = t \ln t + 2t$.

d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.25$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.

Use the Modified Euler method, Runge-Kutta method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.

a. $y' = e^{t-y}$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.5$; actual solution $y(t) = \ln(e^t + e - 1)$.

b. $y' = \dfrac{1+t}{1+y}$, $1 \le t \le 2$, $y(1) = 2$, with $h = 0.5$; actual solution $y(t) = \sqrt{t^2 + 2t + 6} - 1$.

c. $y' = -y + ty^{1/2}$, $2 \le t \le 3$, $y(2) = 2$, with $h = 0.25$; actual solution $y(t) = \left(t - 2 + \sqrt{2}ee^{-t/2}\right)^2$.

d. $y' = t^{-2}(\sin 2t - 2ty)$, $1 \le t \le 2$, $y(1) = 2$, with $h = 0.25$; actual solution $y(t) = \frac{1}{2}t^{-2}(4 + \cos 2 - \cos 2t)$.