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Faculty of Electrical & Electronic Engineering

Lecture:

IMAGE PROCESSING

Chapter 3: Image Transforms

Wavelet Transforms

Nguyen Thanh Hai, PhD

Wavelet Transforms

- Fourier Transform (FT), the analyzing function is the complex exponential $e^{j\omega t}$.
- Short-time FT (STFT), the analyzing function is the complex exponential $e^{j\omega t}$ and $g^*(t-\tau)$
- Wavelet Transform (WT), the analyzing function is a wavelet ψ.
- The WT using operators as shifting and compressing, convoluting or stretching of a wavelet.

Wavelet Transforms

Wavelet Analysis

Wavelet analysis is based on Multiresolution Analysis (MRA) in time-frequency as shown in Fig. 3.7

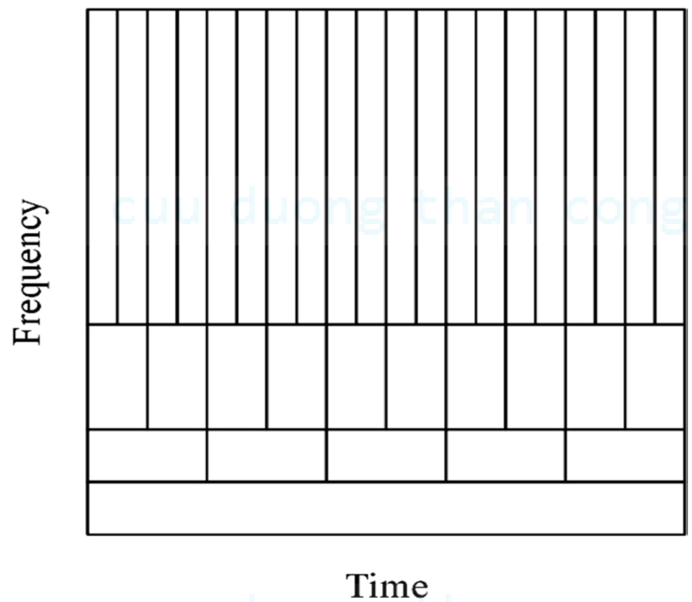

Fig. 3.7. *Analysis of time-frequency with MRA.*

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Wavelet Transforms

- Wavelet analysis is considered between a signal $x(t)$ and Wavelet function $\psi(t)$, in which wavelet $\psi(t)$ is considered as mother wavelet function.

- Wavelet function $\psi(t)$ is a small signal or oscillation for distinguishing different frequencies of the input signal.

- Wavelet contains information of analysis waveform and window size (scale) as shown in Fig. 3.8
- In practice, there are many different wavelet family.

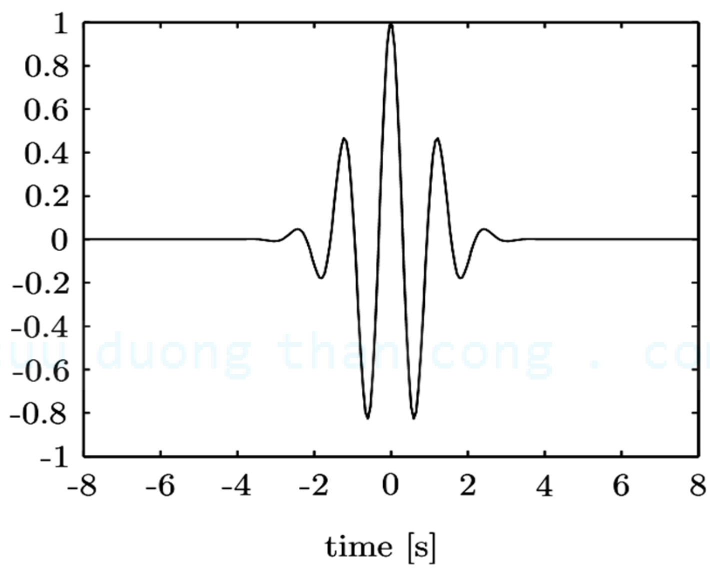

Fig. 3.8. *Morlet Wavelet family*

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Wavelet Transforms

Continuous wavelet transform (CWT) of 1D is defined as follows:

$$C\ (\tau, s) = X_{WT}(\tau, s) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{\infty} x(t)\psi^{*}\left(\frac{t-\tau}{s}\right) dt$$

$X_{WT(\tau,s)}$ is the two-variables function or called the wavelet coefficient $C\ (\tau, s)$, with shifting position τ and scaling parameter s. Mother wavelet function is represented ψ, symbol * is described complex operator.

Inverse Continuous wavelet transform (ICWT) of 1D is defined as follows:

$$x(t) = \frac{1}{C_{\psi}^{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} X_{WT}(\tau, s) \frac{1}{s^{2}} \psi\left(\frac{t-\tau}{s}\right) d\tau ds$$

$$\psi(t) = g(t)e^{-j2\pi f_c t}, g(t) = \sqrt{\pi f_b} e^{t^2/f_b}$$

Image Transforms

- Việc tính toán biến đổi Wavelet thường thực hiện với những giá trị rời rạc ứng với các hệ số tỷ lệ s và độ dịch chuyển τ. Các hệ số Wavelet được gọi là chuỗi Wavelet. Chuỗi Wavelet có thể được tính như sau

$$X_{WT_{m,n}} = \int_{-\infty}^{\infty} x(t)\psi_{m,n}(t)dt$$

$$\psi_{m,n} = s_0^{-m/2}\psi(s_0^{-m}t - n\tau_0)$$

Số nguyên m và n điều chỉnh độ dịch chuyển và độ giản của sóng Wavelet. Ứng với lược đồ nhị nguyên (dyadic grid), $s_0 = 0$ và $\tau_0 = 1$. Các sóng Wavelet này được lựa chọn sao cho trực chuẩn, nghĩa là, chúng trực giao với nhau và được chuẩn hóa để có mức năng lượng đơn vị. Việc lựa chọn này cho phép xây dựng lại tín hiệu gốc thông qua biểu thức rời rạc sau:

$$x(t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} X_{WT_{m,n}} \psi_{m,n}(t)$$

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Image Transforms

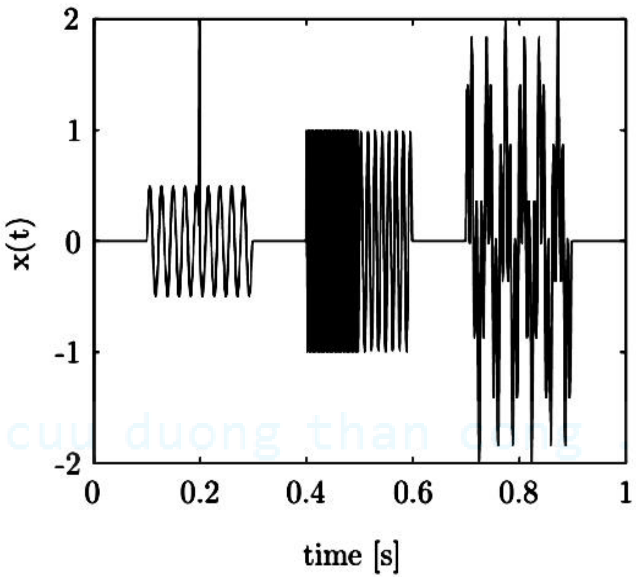

Fig. 3.9. *Input signal with different frequencies and waveforms*

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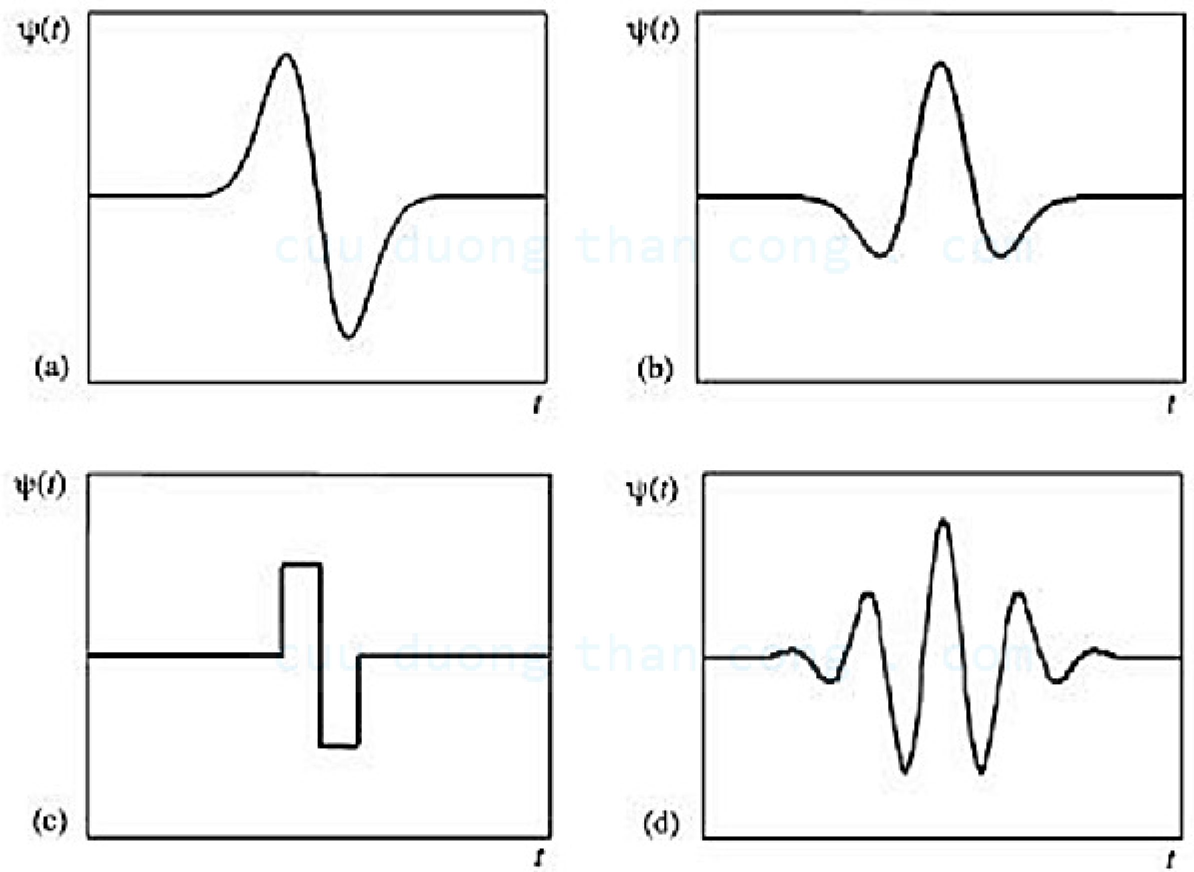
Wavelet Transforms

- Many different wavelets can be used in CWT.
- Depending on signal features to be detected → select a wavelet for analysis easier.

a) Gauss

b) Mexican hat



c) Haar

d) Morlet

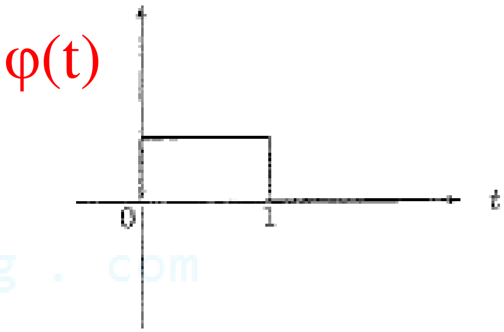
Image Transforms

1D Haar Wavelets

- Haar scaling and wavelet functions:

- Mother scaling function:

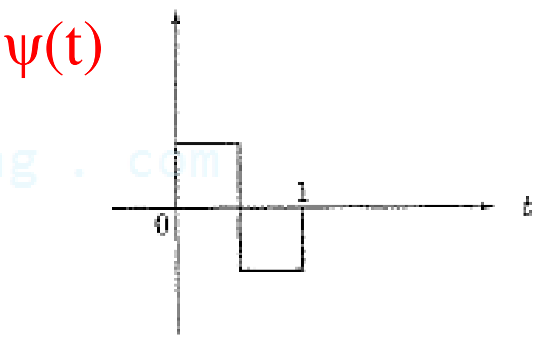
$$\phi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1 \\ 0 & otherwise \end{cases}$$



(a) $\phi(t)$

- Mother wavelet function:

$$\psi(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/2 \\ -1 & \text{if } 1/2 \le x < 1 \\ 0 & otherwise \end{cases}$$



(b) $\psi(t)$

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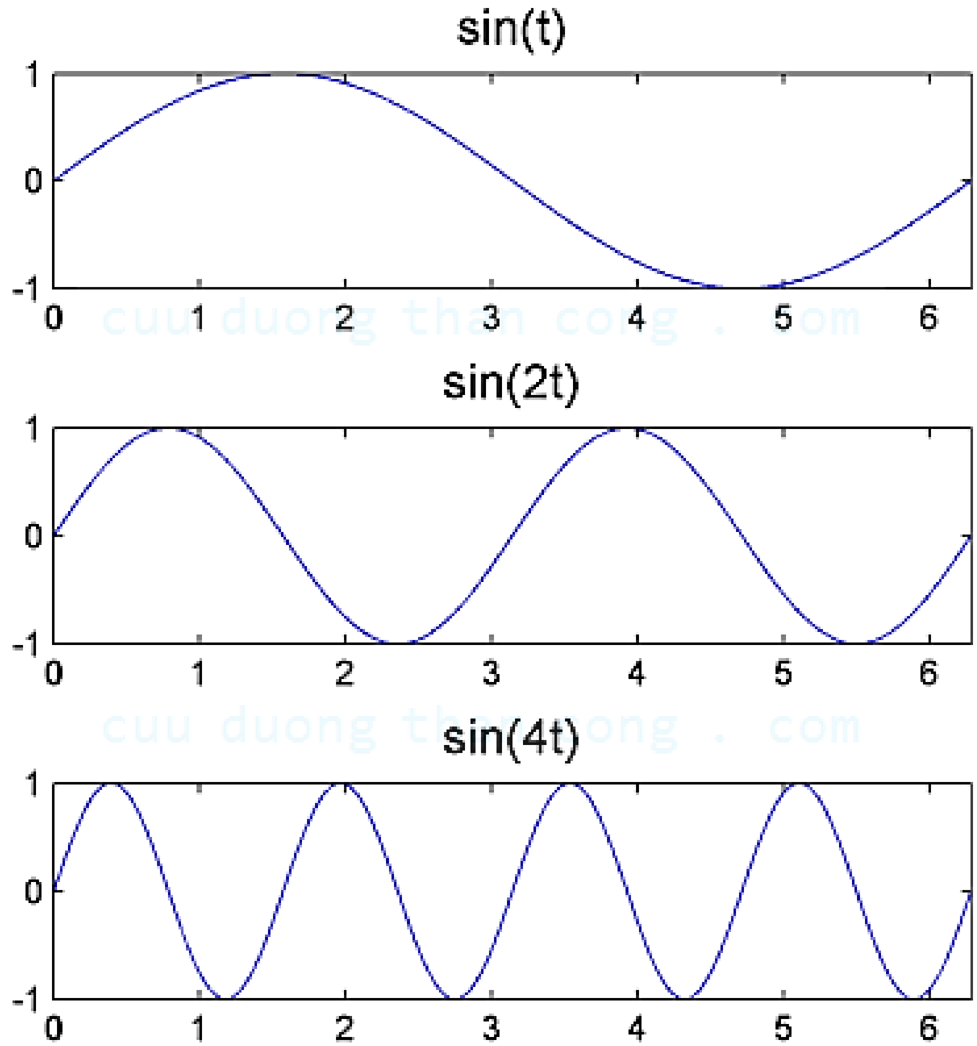
Wavelet Transforms

Continuous wavelet transform (CWT)_Scale

- Example: sine signal with different scales



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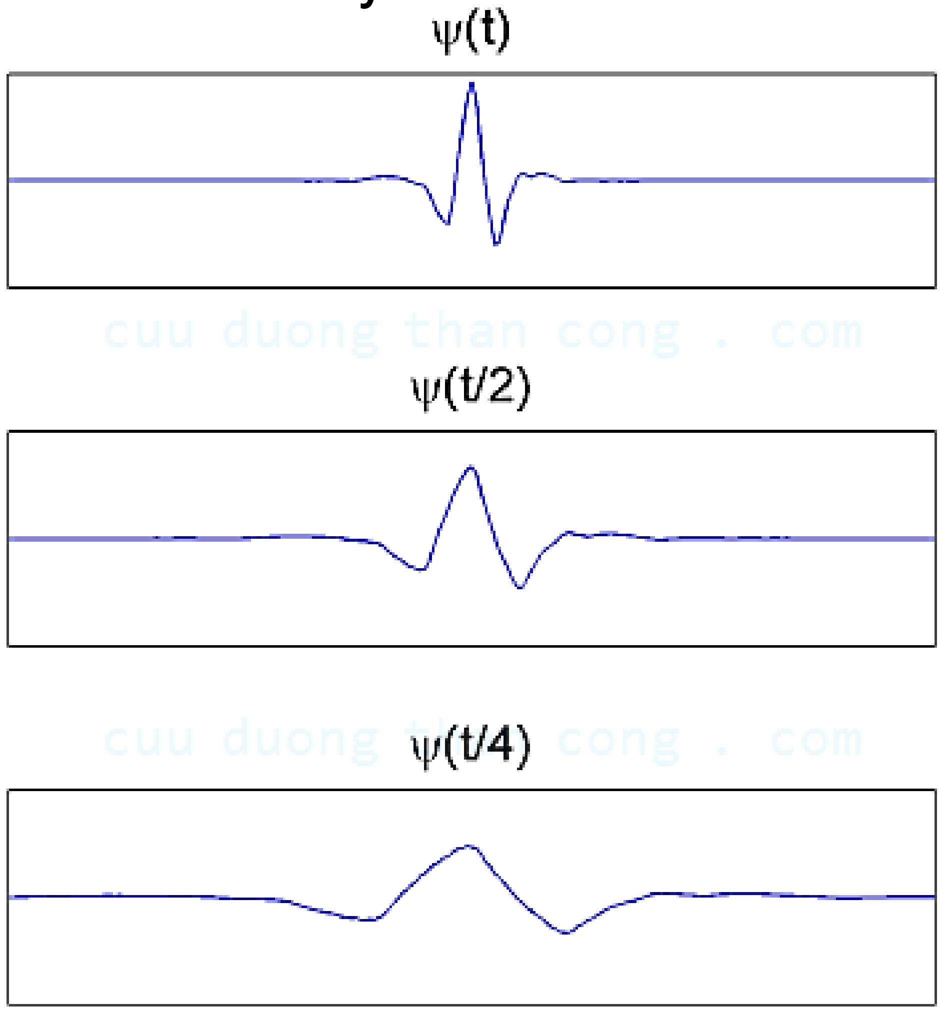
Wavelet Transforms

Continuous wavelet transform (CWT)_Scale

- The scale works exactly the same with wavelet



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Wavelet Transforms

With the 1D discrete signal $f[n]$, the DWT is determined using the following equation:

$$W_\phi(j_0,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\phi_{j_0,k}(n)$$

$$W_\psi(j,k)=\frac{1}{\sqrt{M}}\sum_n f(n)\psi_{j,k}(n) \quad j\ge j_0$$

In which, $\phi_{j_0,k}(n)$ and $\psi_{j,k}(n)$ are functions of scaling and discrete Wavelet determined in the interval $[0, M-1]$ and orthogonal together with j_0 initially

- $W_\phi(j_0,k)$ expresses the approximation coefficient (low frequencies),
- $W_\psi(j,k)$ are the detail coefficients (high frequencies).

The inverse DWT is calculated as follows:

$$f(n)=\frac{1}{\sqrt{M}}\sum_k W_\phi(j_0,k)\phi_{j_0,k}(n)+\frac{1}{\sqrt{M}}\sum_{j=j_0}^{\infty}\sum_k W_\psi(j,k)\psi_{j,k}(n)$$

Wavelet Transforms

Discrete Wavelet transform (DWT) in digital image

This is the 2D wavelet function, including scaling $\phi(x,y)$ and Wavelet $\psi(x,y)$ and calculated as follows:

$$\phi_{j,m,n}(x,y) = 2^{j/2}\phi\left(2^j x - m, 2^j y - n\right),$$

$$\psi^i_{j,m,n}(x,y) = 2^{j/2}\psi^i\left(2^j x - m, 2^j y - n\right), i = \{H, V, D\}$$

3 wavelet functions $\psi^H(x,y)$, $\psi^V(x,y)$, $\psi^D(x,y)$, are still called details; and $\phi A(x,y)$ is called approximation.

In the DWT, one takes care 4 variables

$$\phi(x,y) = \phi(x)\phi(y), \qquad \psi^V(x,y) = \phi(x)\psi(y),$$
$$\psi^H(x,y) = \psi(x)\phi(y), \qquad \psi^D(x,y) = \psi(x)\psi(y),$$

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Wavelet Transforms

Discrete Wavelet transform (DWT) in digital image

Expressions of the approximation and details are described in 2D image as follows:

$$W_{\phi}(j_0,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\,\phi_{j_0,m,n}(x,y)=\{A\}$$

$$W_{\psi}(j,m,n)=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}f(x,y)\,\phi_{j,m,n}^{i}(x,y),i=\{H,V,D\}$$

$$f(x,y)$$
$$=\frac{1}{\sqrt{MN}}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1}W_{\phi}\left(j_{0,m,n}\right)\phi_{j_0,m,n}(x,y)$$
$$+\frac{1}{\sqrt{MN}}\sum_{i=H,V,D}\sum_{j=j_0}^{\infty}\sum_{m}\sum_{n}W_{\psi}^{i}(j,m,n)\,\phi_{j,m,n}^{i}(x,y)$$

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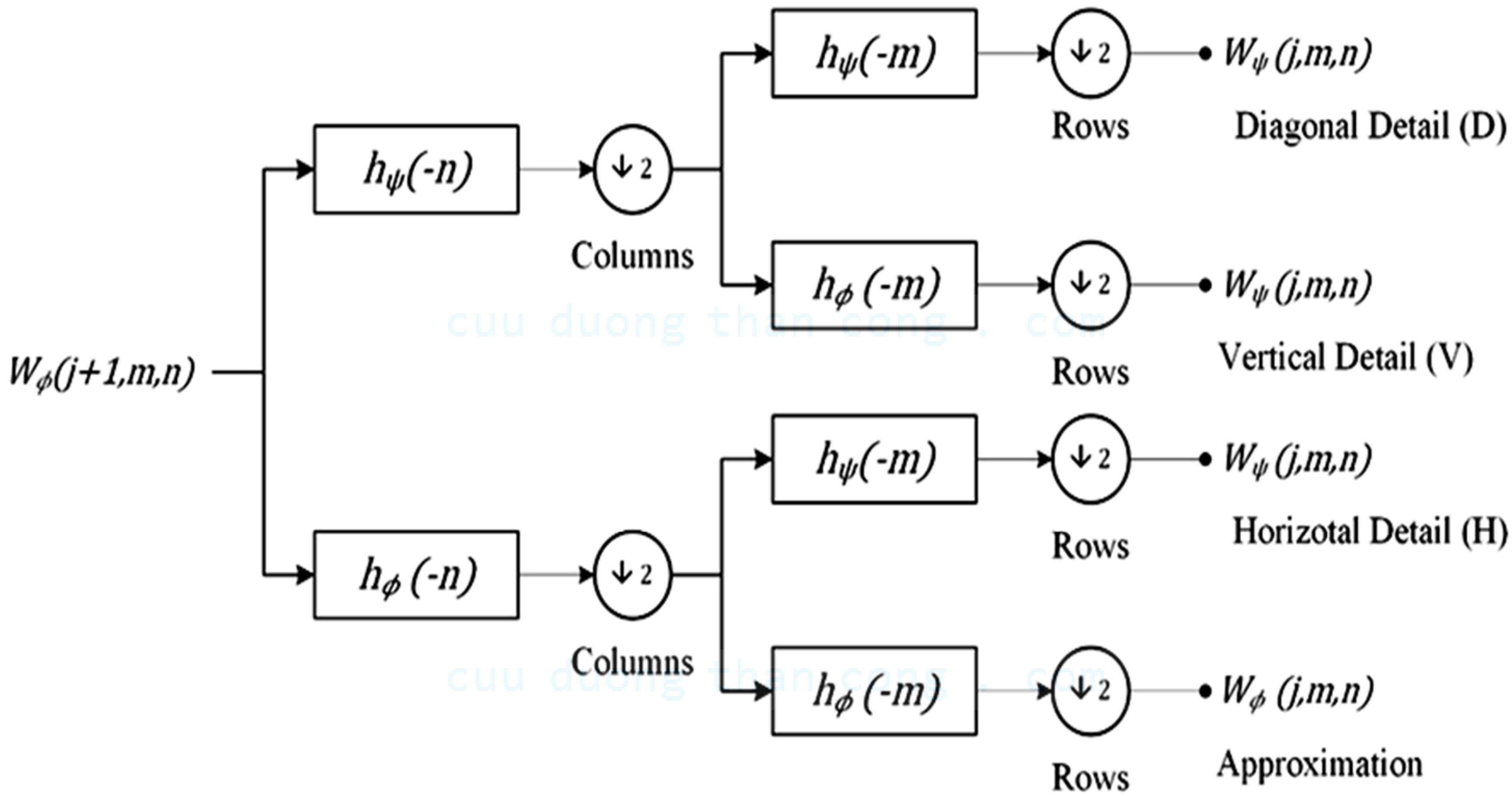
Wavelet Transforms

Diagram of the wavelet transform of an image:



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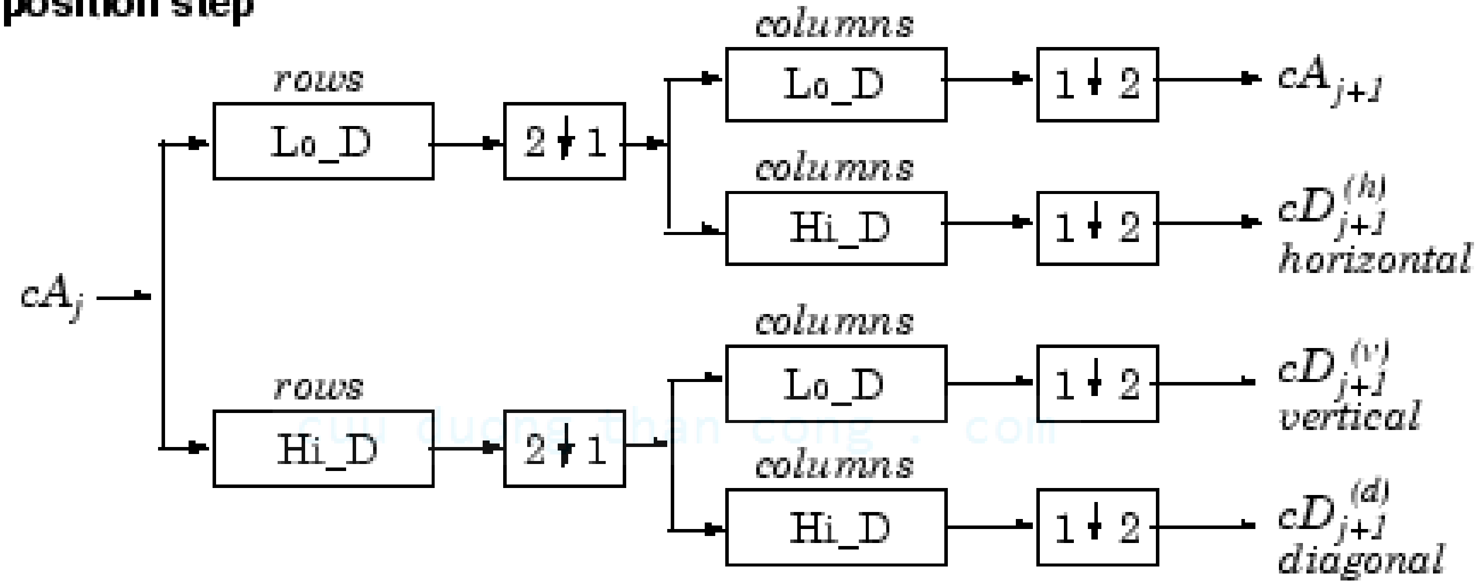
Wavelet Transforms

Two-Dimensional DWT

Decomposition step



where

- 2↓1 : Downsample columns: keep the even indexed columns
- 1↓2 : Downsample rows: keep the even indexed rows
- rows X : Convolve with filter X the rows of the entry
- columns X : Convolve with filter X the columns of the entry

Wavelet Transforms

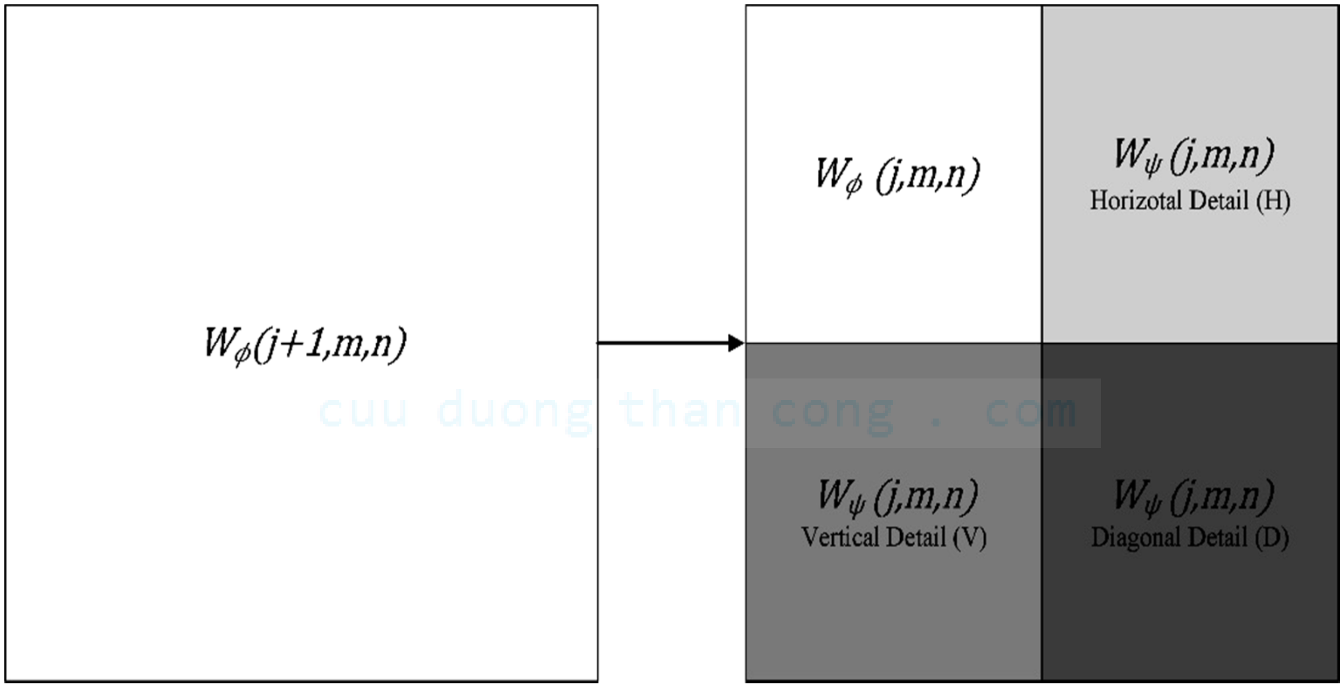


Fig. *Expression of components in an image*

Wavelet Transforms

Some wavelet functions in Matlab:

- [Lo_D,Hi_D,Lo_R,Hi_R] = wfilters('wname');

In which, Lo_D and Hi_D are the lowpass and highpass filters. Lo_R và Hi_R are the lowpass and highpass filters for synthesis, in which the filters must be orthogonal.

- Function wfilters to find types of different wavelets:

[F1,F2] = wfilters('wname','type')

- wavefun :

[phi, psi, xval] = wavefun(vname,iter)

Give approximate vector of phi and psi, and xval is vector ước lượng; iter describes integer numbers

- If variables are orthogonal, we have :

[phi1, psi1, phi2, psi2, xval] = wavefun(wname, iter)

In which phi1 and psi1 are decomposition, phi2 and psi2 are recontruction.

Wavelet Transforms

Ex 3.3: Express filter, scaling and wavelet functions of Haar.

```
clear all;
[Lo_D,Hi_D,Lo_R,Hi_R] =
wfilters('Haar')
waveinfo('Haar')
[phi,psi,xval]=wavefun('Haar',10)
xaxis=zeros(size(xval));
subplot(121);
plot(xval,phi,'k',xval,xaxis,'--k');
axis([0 1 -1.5 1.5]);
axis square;
title('Haar Scaling Function')
subplot(122);
plot(xval,psi,'k',xval,xaxis,'--k');
axis([0 1 -1.5 1.5]);
axis square;
title('Haar Wavelet Function');
```

Results:

```
>>
Lo_D =
    0.7071    0.7071
Hi_D =
   -0.7071    0.7071
Lo_R =
    0.7071    0.7071
Hi_R =
0.7071   -0.7071
```

Wavelet Transforms

Some wavelet functions in Matlab:

wavedec2:

[C, S] = wavedec2(X, N, Lo_D, Hi_D)

In which, X is the 2D matrix, N is the analysis level, Lo_D and Hi_D are the filters. syntax:

[C,S] = wavedec2(X,N,'wname')

Output is row vector C (double) containing Wavelet coefficients and the matrix S (double) determines coefficients in C. The relationship between C and S is described as:

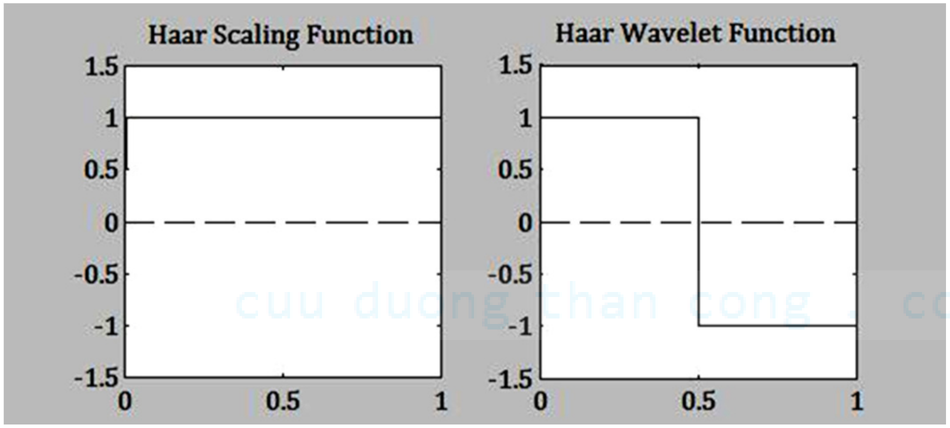


Fig 3.20. *Scaling and Wavelet of Haar family*

Wavelet Transforms

Ex 3.4: Consider wavelet transform using wavedec2 with Haar family

```
clear all;
f=magic(4);
[C,S]=wavedec2(f,1,'Haar');
```

magic(m) function give square matrix with the size of m^2 , having the value of elements from 1 to m^2.

f =

16	2	3	13
5	11	10	8
9	7	6	12
4	14	15	1

C =

Columns 1 through 10

17.0000 17.0000
17.0000 17.0000
1.0000 -1.0000 -1.0000
1.0000 4.0000 -4.0000

Columns 11 through 16

-4.0000 4.0000
10.0000 6.0000 -6.0000
-10.0000

S =

2	2
2	2
4	4

Wavelet Transforms

Ex 3.5: Extract Wavelet coefficients small bands with a typical level

```
clear all;
f=magic(4);
[C,S]=wavedec2(f,2,'Haar');
A=appcoef2(C,S,'Haar',1)
V=detcoef2('v',C,S,2)% vD
H=detcoef2('h',C,S,2)% hD
D=detcoef2('d',C,S,2)% dD
```

In which appcoef2 gives approximation coefficient, and detcoef2 gives detail coefficients; wavedec2 denotes decomposition wavelet with a typical wavelet function.

- A = appcoef2(C,S,'wname',N)
- A = appcoef2(C,S,Lo_R,Hi_R,N) is similar to detcoef2

Wavelet Transforms

Ex 3.6: Calculate DWT with the 4x4 matrix of magic function and Haar family.
The resulat is that

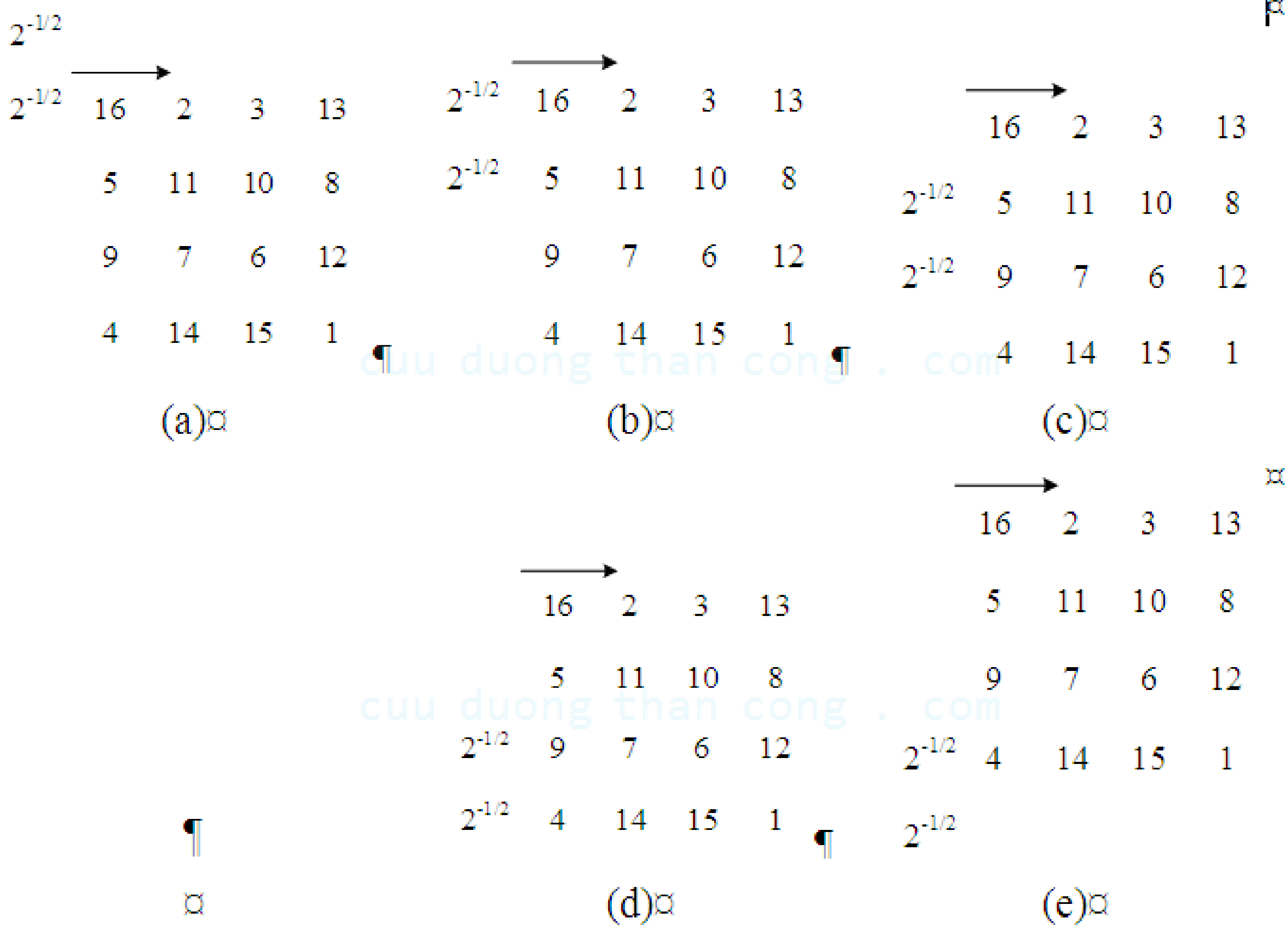
$$f = \begin{bmatrix} 16 & 2 & 3 & 13 \\ 5 & 11 & 10 & 8 \\ 9 & 7 & 6 & 12 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

With the lowpass and highpass filters (orthogonal):
$Lo\_D = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ và $Hi\_D = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

- Approximation components contain coefficients with low frequencies using the lowpass filter.

Wavelet Transforms

Convolution of the image f and the lowpass filter Lo_D cross the row of the matrix.



(a) (b) (c)

(d) (e)

Wavelet Transforms

Bước (a) xác định được các hệ số như sau:

$$O_1(1,1) = 1/\sqrt{2} \times 16$$

$$O_1(1,2) = 1/\sqrt{2} \times 2$$

$$O_1(1,3) = 1/\sqrt{2} \times 3$$

$$O_1(1,4) = 1/\sqrt{2} \times 13$$

Tương tự bước (b) xác định được:

$$O_1(2,1) = 1/\sqrt{2} \times (16 + 5)$$

$$O_1(2,2) = 1/\sqrt{2} \times (2 + 11)$$

$$O_1(2,3) = 1/\sqrt{2} \times (3 + 10)$$

$$O_1(2,4) = 1/\sqrt{2} \times (13 + 8)$$

Ngõ ra O_1 được xác định:

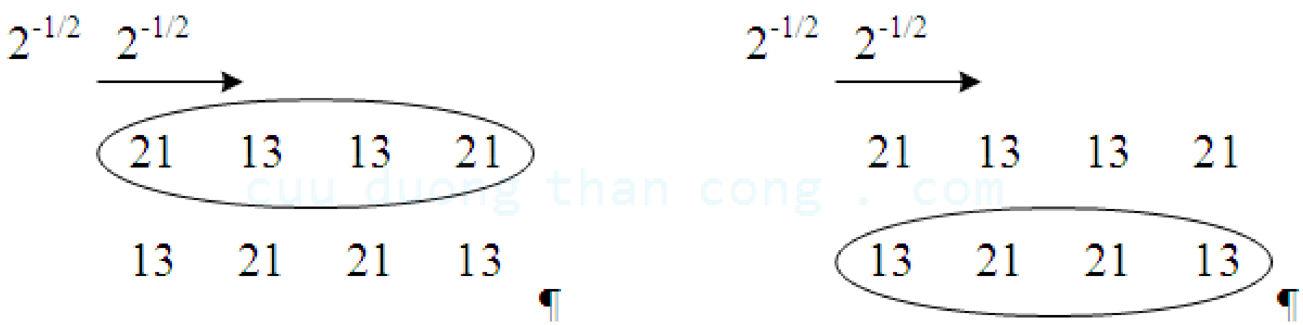
$$O_1 = 1/\sqrt{2} \times \begin{bmatrix} 16 & 2 & 3 & 13 \\ 21 & 13 & 13 & 21 \\ 14 & 18 & 16 & 20 \\ 13 & 21 & 21 & 13 \\ 4 & 14 & 15 & 1 \end{bmatrix}$$

Wavelet Transforms

- Thực hiện giảm mẫu ở những mẫu lẻ theo hàng (loại bỏ hàng lẻ)

$$O_1' = 1/\sqrt{2} \times \begin{bmatrix} 21 & 13 & 13 & 21 \\ 13 & 21 & 21 & 13 \end{bmatrix}$$

- Thực hiện tích chập một lần nữa cũng với hàm bộ lọc thông thấp. Đặt ngõ ra là O_2



(a) (b)

Ngõ ra sau khi tính toán:

$$O_2 = (1/\sqrt{2})^2 \times \begin{bmatrix} 21 & 34 & 26 & 34 & 21 \\ 13 & 34 & 42 & 34 & 13 \end{bmatrix}$$

- Sau đó thực hiện giảm mẫu theo cột với những cột lẻ. Kết quả thu được là các hệ số tần số thấp hay còn gọi là thành phần xấp xỉ.

$$A = (1/\sqrt{2})^2 \times \begin{bmatrix} 34 & 34 \\ 34 & 34 \end{bmatrix} = \begin{bmatrix} 17 & 17 \\ 17 & 17 \end{bmatrix}$$

Wavelet Transforms

$$x = \begin{bmatrix} . \\ x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ . \end{bmatrix} (\downarrow 2) = \begin{bmatrix} . \\ x(0) \\ x(2) \\ x(4) \\ . \end{bmatrix} (\uparrow 2)(\downarrow 2) = \begin{bmatrix} . \\ x(0) \\ 0 \\ x(2) \\ 0 \\ x(4) \\ . \end{bmatrix}$$

Wavelet Transforms

Ex 3.7: Express wavelet with an image

```
clear all;
f=imread('cameraman.bmp');
[cA1,cH1,cV1,cD1]=dwt2(f,'Haar');

f1=[mat2gray(cA1) mat2gray(cH1);mat2gray(cV1)
mat2gray(cD1)];
[cA2,cH2,cV2,cD2]=dwt2(cA1,'Haar');

f22=[mat2gray(cA2) mat2gray(cH2);mat2gray(cV2)
mat2gray(cD2)];

f2=[f22 mat2gray(cH1);mat2gray(cV1) mat2gray(cD1)];
imshow(f1);
figure;imshow(f2);
```

Wavelet transform

Wavelet toolbox : dwt2 (cont)

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Wavelet Transforms

Ex 3.8: Express WT with noise

```
clear all
f=imread('cameraman.bmp');
fn=imnoise(f,'gaussian',0.01);
[cA,cH,cV,cD]=dwt2(fn,'Haar');
[THR_H,SORH_H,KEEPAPP_H]=dde
ncmp('den','wv',cH);
cH=wthresh(cH,SORH_H,THR_H);
[THR_V,SORH_V,KEEPAPP_V]=dden
cmp('den','wv',cV);
cV=wthresh(cH,SORH_V,THR_V);
[THR_D,SORH_D,KEEPAPP_D]=dde
ncmp('den','wv',cD);
cD=wthresh(cH,SORH_D,THR_D);
fdn=uint8(idwt2(cA,cH,cV,cD,'Haar'));
```


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Wavelet Transforms

(a)

Hình 3.24. *Kết quả thực hiện giảm nhiễu sử dụng phân tích Wavelet*

(a) Ảnh gốc

(b) Ảnh bị nhiễu (PSNR=15.01dB)

(c) Ảnh sau khi triệt nhiễu (PSNR=19.51dB)

(b)

(c)

Wavelet Transforms

- Consider functions idwt2, dwt2,

X = idwt2(cA,cH,cV,cD,'wname')

- Image after pepper-salt noise.
- This is the noise with the high frequency affecting details of the image. Thus the algorithm for eliminating the noise used details components with the highpass filter cH, cV and cD.

Wavelet Transforms

Ex: express WT of a 5x5 matrix

- Find DWT of $f=$

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

- Using wavelet Haar, in which: $Lo\_D = \left[\frac{1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$ and $Hi\_D = \left[\frac{-1}{\sqrt{2}} \quad \frac{1}{\sqrt{2}}\right]$

Wavelet Transforms

A =

34.5000	15.0000	31.0000
16.0000	36.5000	25.0000
29.0000	27.0000	18.0000

V =

5.5000	-7.0000	0
-2.0000	-4.5000	0
-7.0000	23.0000	0

H =

6.5000	-6.0000	-1.0000
-6.0000	-3.5000	19.0000
0	0	0

D =

-12.5000	0	0
0.0000	-2.5000	0
0	0	0

Tip: access the 'wavedec2' function root and compare to your answer

Wavelet Transforms

Example:
```
clear all;
f=imread('cameraman.bmp');
[cA1,cH1,cV1,cD1]=dwt2(f,'Haar');
f1=[mat2gray(cA1) mat2gray(cH1);mat2gray(cV1)
mat2gray(cD1)];
```


Wavelet toolbox

[C,S] = wavedec2(X,N,'wname')

Example:
```
clear all;
f=magic(4);
[C,S]=wavedec2(f,1,'Haar');
```

The End