14

Game Theory

Life is full of conflict and competition. Numerous examples involving adversaries in conflict include parlor games, military battles, political campaigns, advertising and marketing campaigns by competing business firms, and so forth. A basic feature in many of these situations is that the final outcome depends primarily upon the combination of strategies selected by the adversaries. Game theory is a mathematical theory that deals with the general features of competitive situations like these in a formal, abstract way. It places particular emphasis on the decision-making processes of the adversaries.

As briefly surveyed in Sec. 14.6, research on game theory continues to delve into rather complicated types of competitive situations. However, the focus in this chapter is on the simplest case, called **two-person, zero-sum games.** As the name implies, these games involve only two adversaries or *players* (who may be armies, teams, firms, and so on). They are called *zero-sum* games because one player wins whatever the other one loses, so that the sum of their net winnings is zero.

Section 14.1 introduces the basic model for two-person, zero-sum games, and the next four sections describe and illustrate different approaches to solving such games. The chapter concludes by mentioning some other kinds of competitive situations that are dealt with by other branches of game theory.

14.1 THE FORMULATION OF TWO-PERSON, ZERO-SUM GAMES

To illustrate the basic characteristics of two-person, zero-sum games, consider the game called *odds and evens.* This game consists simply of each player simultaneously showing either one finger or two fingers. If the number of fingers matches, so that the total number for both players is even, then the player taking evens (say, player 1) wins the bet (say, $1) from the player taking odds (player 2). If the number does not match, player 1 pays $1 to player 2. Thus, each player has two *strategies:* to show either one finger or two fingers. The resulting payoff to player 1 in dollars is shown in the *payoff table* given in Table 14.1.

In general, a two-person game is characterized by

1. The strategies of player 1
2. The strategies of player 2
3. The payoff table

TABLE 14.1 Payoff table for the odds and evens game

Strategy		Player 2: 1	2
Player 1	1	1	−1
	2	−1	1

Before the game begins, each player knows the strategies she or he has available, the ones the opponent has available, and the payoff table. The actual play of the game consists of each player simultaneously choosing a strategy without knowing the opponent's choice.

A strategy may involve only a simple action, such as showing a certain number of fingers in the odds and evens game. On the other hand, in more complicated games involving a series of moves, a **strategy** is a predetermined rule that specifies completely how one intends to respond to each possible circumstance at each stage of the game. For example, a strategy for one side in chess would indicate how to make the next move for *every* possible position on the board, so the total number of possible strategies would be astronomical. Applications of game theory normally involve far less complicated competitive situations than chess does, but the strategies involved can be fairly complex.

The **payoff table** shows the gain (positive or negative) for player 1 that would result from each combination of strategies for the two players. It is given only for player 1 because the table for player 2 is just the negative of this one, due to the zero-sum nature of the game.

The entries in the payoff table may be in any units desired, such as dollars, provided that they accurately represent the *utility* to player 1 of the corresponding outcome. However, utility is not necessarily proportional to the amount of money (or any other commodity) when large quantities are involved. For example, $2 million (after taxes) is probably worth much less than twice as much as $1 million to a poor person. In other words, given the choice between (1) a 50 percent chance of receiving $2 million rather than nothing and (2) being sure of getting $1 million, a poor person probably would much prefer the latter. On the other hand, the outcome corresponding to an entry of 2 in a payoff table should be "worth twice as much" to player 1 as the outcome corresponding to an entry of 1. Thus, given the choice, he or she should be indifferent between a 50 percent chance of receiving the former outcome (rather than nothing) and definitely receiving the latter outcome instead.[1]

A primary objective of game theory is the development of *rational criteria* for selecting a strategy. Two key assumptions are made:

1. *Both* players are *rational.*
2. *Both* players choose their strategies solely to *promote their own welfare* (no compassion for the opponent).

[1]See Sec. 15.5 for a further discussion of the concept of utility.

Game theory contrasts with *decision analysis* (see Chap. 15), where the assumption is that the decision maker is playing a game with a passive opponent—nature—which chooses its strategies in some random fashion.

We shall develop the standard game theory criteria for choosing strategies by means of illustrative examples. In particular, the next section presents a prototype example that illustrates the formulation of a two-person, zero-sum game and its solution in some simple situations. A more complicated variation of this game is then carried into Sec. 14.3 to develop a more general criterion. Sections 14.4 and 14.5 describe a graphical procedure and a linear programming formulation for solving such games.

14.2 SOLVING SIMPLE GAMES—A PROTOTYPE EXAMPLE

Two politicians are running against each other for the U.S. Senate. Campaign plans must now be made for the final 2 days, which are expected to be crucial because of the closeness of the race. Therefore, both politicians want to spend these days campaigning in two key cities, Bigtown and Megalopolis. To avoid wasting campaign time, they plan to travel at night and spend either 1 full day in each city or 2 full days in just one of the cities. However, since the necessary arrangements must be made in advance, neither politician will learn his (or her)[1] opponent's campaign schedule until after he has finalized his own. Therefore, each politician has asked his campaign manager in each of these cities to assess what the impact would be (in terms of votes won or lost) from the various possible combinations of days spent there by himself and by his opponent. He then wishes to use this information to choose his best strategy on how to use these 2 days.

Formulation as a Two-Person, Zero-Sum Game

To formulate this problem as a two-person, zero-sum game, we must identify the two *players* (obviously the two politicians), the *strategies* for each player, and the *payoff table.*

As the problem has been stated, each player has the following three strategies:

Strategy 1 = spend 1 day in each city.
Strategy 2 = spend both days in Bigtown.
Strategy 3 = spend both days in Megalopolis.

By contrast, the strategies would be more complicated in a different situation where each politician learns where his opponent will spend the first day before he finalizes his own plans for his second day. In that case, a typical strategy would be: Spend the first day in Bigtown; if the opponent also spends the first day in Bigtown, then spend the second day in Bigtown; however, if the opponent spends the first day in Megalopolis, then spend the second day in Megalopolis. There would be eight such strategies, one for each combination of the two first-day choices, the opponent's two first-day choices, and the two second-day choices.

Each entry in the payoff table for player 1 represents the *utility* to player 1 (or the negative utility to player 2) of the outcome resulting from the corresponding strategies used by the two players. From the politician's viewpoint, the objective is to *win votes,*

[1]We use only *his* or only *her* in some examples and problems for ease of reading: we do not mean to imply that only men or only women are engaged in the various activities.

TABLE 14.2 Form of the payoff table for politician 1 for the political campaign problem

		Total Net Votes Won by Politician 1 (in Units of 1,000 Votes)		
		Politician 2		
	Strategy	**1**	**2**	**3**
	1			
Politician 1	2			
	3			

and each additional vote (before he learns the outcome of the election) is of equal value to him. Therefore, the appropriate entries for the payoff table for politician 1 are the *total net votes won* from the opponent (i.e., the sum of the net vote changes in the two cities) resulting from these 2 days of campaigning. Using units of 1,000 votes, this formulation is summarized in Table 14.2. Game theory assumes that both players are using the same formulation (including the same payoffs for player 1) for choosing their strategies.

However, we should also point out that this payoff table would *not* be appropriate if additional information were available to the politicians. In particular, assume that they know exactly how the populace is planning to vote 2 days before the election, so that each politician knows exactly how many net votes (positive or negative) he needs to switch in his favor during the last 2 days of campaigning to win the election. Consequently, the only significance of the data prescribed by Table 14.2 would be to indicate which politician would win the election with each combination of strategies. Because the ultimate goal is to win the election and because the size of the plurality is relatively inconsequential, the utility entries in the table then should be some positive constant (say, $+1$) when politician 1 wins and -1 when he loses. Even if only a *probability* of winning can be determined for each combination of strategies, the appropriate entries would be the probability of winning minus the probability of losing because they then would represent *expected* utilities. However, sufficiently accurate data to make such determinations usually are not available, so this example uses the thousands of total net votes won by politician 1 as the entries in the payoff table.

Using the form given in Table 14.2, we give three alternative sets of data for the payoff table to illustrate how to solve three different kinds of games.

Variation 1 of the Example

Given that Table 14.3 is the payoff table for player 1 (politician 1), which strategy should each player select?

This situation is a rather special one, where the answer can be obtained just by applying the concept of **dominated strategies** to rule out a succession of inferior strategies until only one choice remains.

TABLE 14.3 Payoff table for player 1 for variation 1 of the political campaign problem

		Player 2		
	Strategy	**1**	**2**	**3**
	1	1	2	4
Player 1	2	1	0	5
	3	0	1	−1

A strategy is **dominated** by a second strategy if the second strategy is *always at least as good* (and sometimes better) regardless of what the opponent does. A dominated strategy can be eliminated immediately from further consideration.

At the outset, Table 14.3 includes no dominated strategies for player 2. However, for player 1, strategy 3 is dominated by strategy 1 because the latter has larger payoffs ($1 > 0, 2 > 1, 4 > -1$) regardless of what player 2 does. Eliminating strategy 3 from further consideration yields the following reduced payoff table:

	1	2	3
1	1	2	4
2	1	0	5

Because both players are assumed to be rational, player 2 also can deduce that player 1 has only these two strategies remaining under consideration. Therefore, player 2 now *does* have a dominated strategy—strategy 3, which is dominated by both strategies 1 and 2 because they always have smaller losses for player 2 (payoffs to player 1) in this reduced payoff table (for strategy 1: $1 < 4$, $1 < 5$; for strategy 2: $2 < 4$, $0 < 5$). Eliminating this strategy yields

	1	2
1	1	2
2	1	0

At this point, strategy 2 for player 1 becomes dominated by strategy 1 because the latter is better in column 2 ($2 > 0$) and equally good in column 1 ($1 = 1$). Eliminating the dominated strategy leads to

	1	2
1	1	2

Strategy 2 for player 2 now is dominated by strategy 1 ($1 < 2$), so strategy 2 should be eliminated.

Consequently, both players should select their strategy 1. Player 1 then will receive a payoff of 1 from player 2 (that is, politician 1 will gain 1,000 votes from politician 2).

In general, the payoff to player 1 when both players play optimally is referred to as the **value of the game.** A game that has a value of 0 is said to be a **fair game.** Since this particular game has a value of 1, it is *not* a fair game.

The concept of a dominated strategy is a very useful one for reducing the size of the payoff table that needs to be considered and, in unusual cases like this one, actually identifying the optimal solution for the game. However, most games require another approach to at least finish solving, as illustrated by the next two variations of the example.

Variation 2 of the Example

Now suppose that the current data give Table 14.4 as the payoff table for player 1 (politician 1). This game does not have dominated strategies, so it is not obvious what the players should do. What line of reasoning does game theory say they should use?

Consider player 1. By selecting strategy 1, he could win 6 or could lose as much as 3. However, because player 2 is rational and thus will seek a strategy that will protect himself from large payoffs to player 1, it seems likely that player 1 would incur a loss by playing strategy 1. Similarly, by selecting strategy 3, player 1 could win 5, but more probably his rational opponent would avoid this loss and instead administer a loss to player 1 which could be as large as 4. On the other hand, if player 1 selects strategy 2, he is guaranteed not to lose anything and he could even win something. Therefore, because it provides the *best guarantee* (a payoff of 0), strategy 2 seems to be a "rational" choice for player 1 against his rational opponent. (This line of reasoning assumes that both players are averse to risking larger losses than necessary, in contrast to those individuals who enjoy gambling for a large payoff against long odds.)

Now consider player 2. He could lose as much as 5 or 6 by using strategy 1 or 3, but is guaranteed at least breaking even with strategy 2. Therefore, by the same reasoning of seeking the best guarantee against a rational opponent, his apparent choice is strategy 2.

If both players choose their strategy 2, the result is that both break even. Thus, in this case, neither player improves upon his best guarantee, but both also are forcing the opponent into the same position. Even when the opponent deduces a player's strategy, the opponent cannot exploit this information to improve his position. Stalemate.

TABLE 14.4 Payoff table for player 1 for variation 2 of the political campaign problem

		Player 2			
	Strategy	**1**	**2**	**3**	**Minimum**
	1	−3	−2	6	−3
Player 1	2	2	0	2	0 ← Maximin value
	3	5	−2	−4	−4
	Maximum:	5	0 ↑ Minimax value	6	

The end product of this line of reasoning is that each player should play in such a way as to *minimize his maximum losses* whenever the resulting choice of strategy cannot be exploited by the opponent to then improve his position. This so-called **minimax criterion** is a standard criterion proposed by game theory for selecting a strategy. In effect, this criterion says to select a strategy that would be best even if the selection were being announced to the opponent before the opponent chooses a strategy. In terms of the payoff table, it implies that *player 1* should select the strategy whose *minimum payoff* is *largest,* whereas *player 2* should choose the one whose *maximum payoff to player 1* is the *smallest.* This criterion is illustrated in Table 14.4, where strategy 2 is identified as the *maximin strategy* for player 1 and strategy 2 is the *minimax strategy* for player 2. The resulting payoff of 0 is the value of the game, so this is a fair game.

Notice the interesting fact that the same entry in this payoff table yields both the maximin and minimax values. The reason is that this entry is both the minimum in its row and the maximum of its column. The position of any such entry is called a **saddle point.**

The fact that this game possesses a saddle point was actually crucial in determining how it should be played. Because of the saddle point, neither player can take advantage of the opponent's strategy to improve his own position. In particular, when player 2 predicts or learns that player 1 is using strategy 2, player 2 would incur a loss instead of breaking even if he were to change from his original plan of using his strategy 2. Similarly, player 1 would only worsen his position if he were to change his plan. Thus, neither player has any motive to consider changing strategies, either to take advantage of his opponent or to prevent the opponent from taking advantage of him. Therefore, since this is a **stable solution** (also called an *equilibrium solution*), players 1 and 2 should exclusively use their maximin and minimax strategies, respectively.

As the next variation illustrates, some games do not possess a saddle point, in which case a more complicated analysis is required.

Variation 3 of the Example

Late developments in the campaign result in the final payoff table for player 1 (politician 1) given by Table 14.5. How should this game be played?

Suppose that both players attempt to apply the minimax criterion in the same way as in variation 2. Player 1 can guarantee that he will lose no more than 2 by playing strategy 1. Similarly, player 2 can guarantee that he will lose no more than 2 by playing strategy 3.

TABLE 14.5 Payoff table for player 1 for variation 3 of the political campaign problem

		Player 2			
Strategy		**1**	**2**	**3**	**Minimum**
	1	0	−2	2	−2 ← Maximin value
Player 1	2	5	4	−3	−3
	3	2	3	−4	−4
	Maximum:	5	4	2 ↑ Minimax value	

However, notice that the maximin value (−2) and the minimax value (2) do not coincide in this case. The result is that there is *no saddle point.*

What are the resulting consequences if both players plan to use the strategies just derived? It can be seen that player 1 would win 2 from player 2, which would make player 2 unhappy. Because player 2 is rational and can therefore foresee this outcome, he would then conclude that he can do much better, actually winning 2 rather than losing 2, by playing strategy 2 instead. Because player 1 is also rational, he would anticipate this switch and conclude that he can improve considerably, from −2 to 4, by changing to strategy 2. Realizing this, player 2 would then consider switching back to strategy 3 to convert a loss of 4 to a gain of 3. This possibility of a switch would cause player 1 to consider again using strategy 1, after which the whole cycle would start over again. Therefore, even though this game is being played only once, *any* tentative choice of a strategy leaves that player with a motive to consider changing strategies, either to take advantage of his opponent or to prevent the opponent from taking advantage of him.

In short, the originally suggested solution (player 1 to play strategy 1 and player 2 to play strategy 3) is an **unstable solution,** so it is necessary to develop a more satisfactory solution. But what kind of solution should it be?

The key fact seems to be that whenever one player's strategy is predictable, the opponent can take advantage of this information to improve his position. Therefore, an essential feature of a rational plan for playing a game such as this one is that neither player should be able to deduce which strategy the other will use. Hence, in this case, rather than applying some known criterion for determining a single strategy that will definitely be used, it is necessary to choose among alternative acceptable strategies on some kind of random basis. By doing this, neither player knows in advance which of his own strategies will be used, let alone what his opponent will do.

This suggests, in very general terms, the kind of approach that is required for games lacking a saddle point. In the next section we discuss the approach more fully. Given this foundation, the following two sections will develop procedures for finding an optimal way of playing such games. This particular variation of the political campaign problem will continue to be used to illustrate these ideas as they are developed.

14.3 GAMES WITH MIXED STRATEGIES

Whenever a game does not possess a saddle point, game theory advises each player to assign a probability distribution over her set of strategies. To express this mathematically, let

x_i = probability that player 1 will use strategy i ($i = 1, 2, \dots, m$),
y_j = probability that player 2 will use strategy j ($j = 1, 2, \dots, n$),

where m and n are the respective numbers of available strategies. Thus, player 1 would specify her plan for playing the game by assigning values to $x_1, x_2, \dots, x_m$. Because these values are probabilities, they would need to be nonnegative and add to 1. Similarly, the plan for player 2 would be described by the values she assigns to her decision variables $y_1, y_2, \dots, y_n$. These plans $(x_1, x_2, \dots, x_m)$ and $(y_1, y_2, \dots, y_n)$ are usually referred to as **mixed strategies,** and the original strategies are then called **pure strategies.**

When the game is actually played, it is necessary for each player to use one of her pure strategies. However, this pure strategy would be chosen by using some random device to obtain a random observation from the probability distribution specified by the mixed strategy, where this observation would indicate which particular pure strategy to use.

To illustrate, suppose that players 1 and 2 in variation 3 of the political campaign problem (see Table 14.5) select the mixed strategies $(x_1, x_2, x_3) = (\frac{1}{2}, \frac{1}{2}, 0)$ and $(y_1, y_2, y_3) = (0, \frac{1}{2}, \frac{1}{2})$, respectively. This selection would say that player 1 is giving an equal chance (probability of $\frac{1}{2}$) of choosing either (pure) strategy 1 or 2, but he is discarding strategy 3 entirely. Similarly, player 2 is randomly choosing between his last two pure strategies. To play the game, each player could then flip a coin to determine which of his two acceptable pure strategies he will actually use.

Although no completely satisfactory measure of performance is available for evaluating mixed strategies, a very useful one is the *expected payoff.* By applying the probability theory definition of expected value, this quantity is

$$\text{Expected payoff for player 1} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j,$$

where p_{ij} is the payoff if player 1 uses pure strategy i and player 2 uses pure strategy j. In the example of mixed strategies just given, there are four possible payoffs $(-2, 2, 4, -3)$, each occurring with a probability of $\frac{1}{4}$, so the expected payoff is $\frac{1}{4}(-2 + 2 + 4 - 3) = \frac{1}{4}$. Thus, this measure of performance does not disclose anything about the risks involved in playing the game, but it does indicate what the average payoff will tend to be if the game is played many times.

By using this measure, game theory extends the concept of the minimax criterion to games that lack a saddle point and thus need mixed strategies. In this context, the **minimax criterion** says that a given player should select the mixed strategy that *minimizes* the *maximum expected loss* to himself. Equivalently, when we focus on payoffs (player 1) rather than losses (player 2), this criterion says to *maximin* instead, i.e., *maximize* the *minimum expected payoff* to the player. By the *minimum expected payoff* we mean the smallest possible expected payoff that can result from any mixed strategy with which the opponent can counter. Thus, the mixed strategy for player 1 that is *optimal* according to this criterion is the one that provides the *guarantee* (minimum expected payoff) that is *best* (maximal). (The value of this best guarantee is the *maximin value,* denoted by $\underline{v}$.) Similarly, the *optimal* strategy for player 2 is the one that provides the *best guarantee,* where *best* now means *minimal* and *guarantee* refers to the *maximum expected loss* that can be administered by any of the opponent's mixed strategies. (This best guarantee is the *minimax value,* denoted by $\overline{v}$.)

Recall that when only pure strategies were used, games not having a saddle point turned out to be *unstable* (no stable solutions). The reason was essentially that $\underline{v} < \overline{v}$, so that the players would want to change their strategies to improve their positions. Similarly, for games with mixed strategies, it is necessary that $\underline{v} = \overline{v}$ for the optimal solution to be *stable.* Fortunately, according to the minimax theorem of game theory, this condition always holds for such games.

Minimax theorem: If mixed strategies are allowed, the pair of mixed strategies that is optimal according to the minimax criterion provides a *stable solution* with $\underline{v} = \overline{v} = v$ (the value of the game), so that neither player can do better by unilaterally changing her or his strategy.

One proof of this theorem is included in Sec. 14.5.

Although the concept of mixed strategies becomes quite intuitive if the game is played *repeatedly,* it requires some interpretation when the game is to be played just *once.* In this case, using a mixed strategy still involves selecting and using *one* pure strategy (randomly selected from the specified probability distribution), so it might seem more sensible to ignore this randomization process and just choose the one "best" pure strategy to be used. However, we have already illustrated for variation 3 in the preceding section that a player must *not* allow the opponent to deduce what his strategy will be (i.e., the solution procedure under the rules of game theory must not *definitely* identify which pure strategy will be used when the game is unstable). Furthermore, even if the opponent is able to use only his knowledge of the tendencies of the first player to deduce probabilities (for the pure strategy chosen) that are different from those for the optimal mixed strategy, then the opponent still can take advantage of this knowledge to reduce the expected payoff to the first player. Therefore, the only way to guarantee attaining the optimal expected payoff v is to randomly select the pure strategy to be used from the probability distribution for the optimal mixed strategy. (Valid statistical procedures for making such a random selection are discussed in Sec. 22.4.)

Now we need to show how to find the optimal mixed strategy for each player. There are several methods of doing this. One is a graphical procedure that may be used whenever one of the players has only two (undominated) pure strategies; this approach is described in the next section. When larger games are involved, the usual method is to transform the problem to a linear programming problem that then can be solved by the simplex method on a computer; Sec. 14.5 discusses this approach.

14.4 GRAPHICAL SOLUTION PROCEDURE

Consider any game with mixed strategies such that, after dominated strategies are eliminated, one of the players has only two pure strategies. To be specific, let this player be player 1. Because her mixed strategies are (x_1, x_2) and $x_2 = 1 - x_1$, it is necessary for her to solve only for the optimal value of x_1. However, it is straightforward to plot the expected payoff as a function of x_1 for each of her opponent's pure strategies. This graph can then be used to identify the point that maximizes the minimum expected payoff. The opponent's minimax mixed strategy can also be identified from the graph.

To illustrate this procedure, consider variation 3 of the political campaign problem (see Table 14.5). Notice that the third pure strategy for player 1 is dominated by her second, so the payoff table can be reduced to the form given in Table 14.6. Therefore, for

TABLE 14.6 Reduced payoff table for player 1 for variation 3 of the political campaign problem

			Player 2		
		Probability	y_1	y_2	y_3
	Probability	**Pure Strategy**	**1**	**2**	**3**
Player 1	x_1	1	0	−2	2
	$1 - x_1$	2	5	4	−3

each of the pure strategies available to player 2, the expected payoff for player 1 will be

(y_1, y_2, y_3)	Expected Payoff
(1, 0, 0)	$0x_1 + 5(1 - x_1) = 5 - 5x_1$
(0, 1, 0)	$-2x_1 + 4(1 - x_1) = 4 - 6x_1$
(0, 0, 1)	$2x_1 - 3(1 - x_1) = -3 + 5x_1$

Now plot these expected-payoff lines on a graph, as shown in Fig. 14.1. For any given values of x_1 and (y_1, y_2, y_3), the expected payoff will be the appropriate weighted average of the corresponding points on these three lines. In particular,

$$\text{Expected payoff for player } 1 = y_1(5 - 5x_1) + y_2(4 - 6x_1) + y_3(-3 + 5x_1).$$

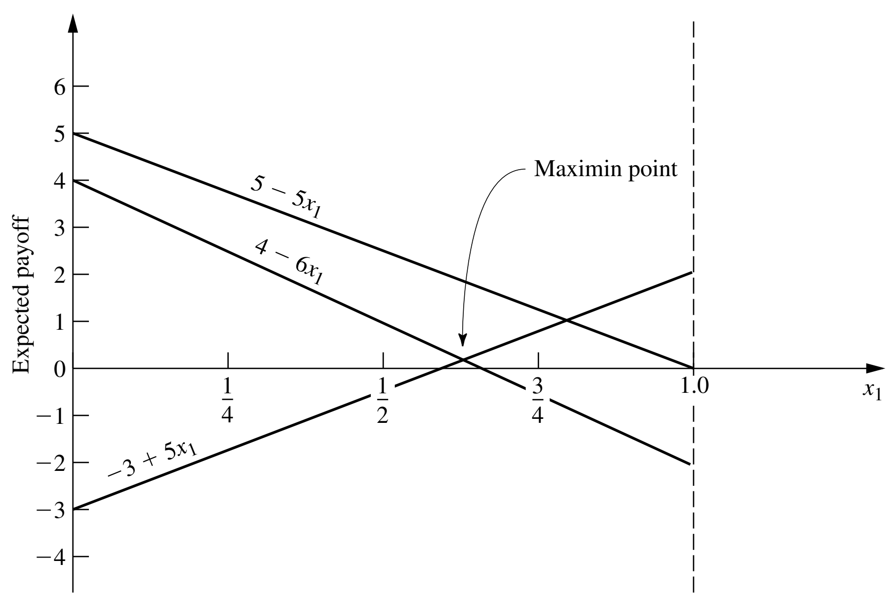
Remember that player 2 wants to minimize this expected payoff for player 1. Given x_1, player 2 can minimize this expected payoff by choosing the pure strategy that corresponds to the "bottom" line for that x_1 in Fig. 14.1 (either $-3 + 5x_1$ or $4 - 6x_1$, but never $5 - 5x_1$). According to the minimax (or maximin) criterion, player 1 wants to maximize this minimum expected payoff. Consequently, player 1 should select the value of x_1 where the bottom line peaks, i.e., where the $(-3 + 5x_1)$ and $(4 - 6x_1)$ lines intersect, which yields an expected payoff of

$$\underline{v} = v = \max_{0 \le x_1 \le 1} \{\min\{-3 + 5x_1, 4 - 6x_1\}\}.$$

To solve algebraically for this optimal value of x_1 at the intersection of the two lines $-3 + 5x_1$ and $4 - 6x_1$, we set

$$-3 + 5x_1 = 4 - 6x_1,$$

FIGURE 14.1
Graphical procedure for solving games



which yields $x_1 = \frac{7}{11}$. Thus, $(x_1, x_2) = (\frac{7}{11}, \frac{4}{11})$ is the *optimal mixed strategy* for player 1, and

$$\underline{v} = v = -3 + 5\left(\frac{7}{11}\right) = \frac{2}{11}$$

is the value of the game.

To find the corresponding optimal mixed strategy for player 2, we now reason as follows. According to the definition of the minimax value $\bar{v}$ and the minimax theorem, the expected payoff resulting from the optimal strategy $(y_1, y_2, y_3) = (y_1^*, y_2^*, y_3^*)$ will satisfy the condition

$$y_1^*(5 - 5x_1) + y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) \le \bar{v} = v = \frac{2}{11}$$

for all values of x_1 $(0 \le x_1 \le 1)$. Furthermore, when player 1 is playing optimally (that is, $x_1 = \frac{7}{11}$), this inequality will be an equality (by the minimax theorem), so that

$$\frac{20}{11}y_1^* + \frac{2}{11}y_2^* + \frac{2}{11}y_3^* = v = \frac{2}{11}.$$

Because (y_1, y_2, y_3) is a probability distribution, it is also known that

$$y_1^* + y_2^* + y_3^* = 1.$$

Therefore, $y_1^* = 0$ because $y_1^* > 0$ would violate the next-to-last equation; i.e., the expected payoff on the graph at $x_1 = \frac{7}{11}$ would be above the maximin point. (In general, any line that does not pass through the maximin point must be given a zero weight to avoid increasing the expected payoff above this point.)

Hence,

$$y_2^*\,(4 - 6x_1) + y_3^*\,(-3 + 5x_1) \begin{cases} \le \dfrac{2}{11} & \text{for } 0 \le x_1 \le 1, \\ = \dfrac{2}{11} & \text{for } x_1 = \dfrac{7}{11}. \end{cases}$$

But y_2^* and y_3^* are numbers, so the left-hand side is the equation of a straight line, which is a fixed weighted average of the two "bottom" lines on the graph. Because the ordinate of this line must equal $\frac{2}{11}$ at $x_1 = \frac{7}{11}$, and because it must never exceed $\frac{2}{11}$, the line necessarily is horizontal. (This conclusion is always true unless the optimal value of x_1 is either 0 or 1, in which case player 2 also should use a single pure strategy.) Therefore,

$$y_2^*(4 - 6x_1) + y_3^*(-3 + 5x_1) = \frac{2}{11}, \qquad \text{for } 0 \le x_1 \le 1.$$

Hence, to solve for y_2^* and y_3^*, select two values of x_1 (say, 0 and 1), and solve the resulting two simultaneous equations. Thus,

$$4y_2^* - 3y_3^* = \frac{2}{11},$$

$$-2y_2^* + 2y_3^* = \frac{2}{11},$$

which has a simultaneous solution of $y_2^* = \frac{5}{11}$ and $y_3^* = \frac{6}{11}$. Therefore, the *optimal mixed strategy* for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$.

If, in another problem, there should happen to be more than two lines passing through the maximin point, so that more than two of the y_j^* values can be greater than zero, this condition would imply that there are many ties for the optimal mixed strategy for player 2. One such strategy can then be identified by setting all but two of these y_j^* values equal to zero and solving for the remaining two in the manner just described. For the remaining two, the associated lines must have positive slope in one case and negative slope in the other.

Although this graphical procedure has been illustrated for only one particular problem, essentially the same reasoning can be used to solve any game with mixed strategies that has only two undominated pure strategies for one of the players.

14.5 SOLVING BY LINEAR PROGRAMMING

Any game with mixed strategies can be solved by transforming the problem to a linear programming problem. As you will see, this transformation requires little more than applying the minimax theorem and using the definitions of the maximin value $\underline{v}$ and minimax value $\overline{v}$.

First, consider how to find the optimal mixed strategy for player 1. As indicated in Sec. 14.3,

$$\text{Expected payoff for player 1} = \sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j$$

and the strategy $(x_1, x_2, \ldots, x_m)$ is optimal if

$$\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} x_i y_j \geq \underline{v} = v$$

for all opposing strategies $(y_1, y_2, \ldots, y_n)$. Thus, this inequality will need to hold, e.g., for each of the pure strategies of player 2, that is, for each of the strategies $(y_1, y_2, \ldots, y_n)$ where one $y_j = 1$ and the rest equal 0. Substituting these values into the inequality yields

$$\sum_{i=1}^{m} p_{ij} x_i \geq v \qquad \text{for } j = 1, 2, \ldots, n,$$

so that the inequality *implies* this set of n inequalities. Furthermore, this set of n inequalities *implies* the original inequality (rewritten)

$$\sum_{j=1}^{n} y_j \left(\sum_{i=1}^{m} p_{ij} x_i \right) \geq \sum_{j=1}^{n} y_j v = v,$$

since

$$\sum_{j=1}^{n} y_j = 1.$$

Because the implication goes in both directions, it follows that imposing this set of n linear inequalities is *equivalent* to requiring the original inequality to hold for all strategies

$(y_1, y_2, \ldots, y_n)$. But these n inequalities are legitimate linear programming constraints, as are the additional constraints

$$x_1 + x_2 + \cdots + x_m = 1$$
$$x_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m$$

that are required to ensure that the x_i are probabilities. Therefore, any solution $(x_1, x_2, \ldots, x_m)$ that satisfies this entire set of linear programming constraints is the desired optimal mixed strategy.

Consequently, the problem of finding an optimal mixed strategy has been reduced to finding a feasible solution for a linear programming problem, which can be done as described in Chap. 4. The two remaining difficulties are that (1) v is unknown and (2) the linear programming problem has no objective function. Fortunately, both these difficulties can be resolved at one stroke by replacing the unknown constant v by the variable x_{m+1} and then *maximizing* x_{m+1}, so that x_{m+1} automatically will equal v (by definition) at the *optimal* solution for the linear programming problem!

To summarize, player 1 would find his optimal mixed strategy by using the simplex method to solve the linear programming problem:

$$\text{Maximize} \qquad x_{m+1},$$

subject to

$$\begin{aligned} p_{11}x_1 + p_{21}x_2 + \cdots + p_{m1}x_m - x_{m+1} &\geq 0 \\ p_{12}x_1 + p_{22}x_2 + \cdots + p_{m2}x_m - x_{m+1} &\geq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{1n}x_1 + p_{2n}x_2 + \cdots + p_{mn}x_m - x_{m+1} &\geq 0 \\ x_1 + x_2 + \cdots + x_m &= 1 \end{aligned}$$

and

$$x_i \geq 0, \qquad \text{for } i = 1, 2, \ldots, m.$$

Note that x_{m+1} is not restricted to be nonnegative, whereas the simplex method can be applied only after *all* the variables have nonnegativity constraints. However, this matter can be easily rectified, as will be discussed shortly.

Now consider player 2. He could find his optimal mixed strategy by rewriting the payoff table as the payoff to himself rather than to player 1 and then by proceeding exactly as just described. However, it is enlightening to summarize his formulation in terms of the original payoff table. By proceeding in a way that is completely analogous to that just described, player 2 would conclude that his optimal mixed strategy is given by an optimal solution to the linear programming problem:

$$\text{Minimize} \qquad y_{n+1},$$

subject to

$$\begin{aligned} p_{11}y_1 + p_{12}y_2 + \cdots + p_{1n}y_n - y_{n+1} &\leq 0 \\ p_{21}y_1 + p_{22}y_2 + \cdots + p_{2n}y_n - y_{n+1} &\leq 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{m1}y_1 + p_{m2}y_2 + \cdots + p_{mn}y_n - y_{n+1} &\leq 0 \\ y_1 + y_2 + \cdots + y_n &= 1 \end{aligned}$$

and

$$y_j \geq 0, \qquad \text{for } j = 1, 2, \ldots, n.$$

It is easy to show (see Prob. 14.5-5 and its hint) that this linear programming problem and the one given for player 1 are *dual* to each other in the sense described in Secs. 6.1 and 6.4. This fact has several important implications. One implication is that the optimal mixed strategies for both players can be found by solving only one of the linear programming problems because the optimal dual solution is an automatic by-product of the simplex method calculations to find the optimal primal solution. A second implication is that this brings all *duality theory* (described in Chap. 6) to bear upon the interpretation and analysis of games.

A related implication is that this provides a simple proof of the minimax theorem. Let x^*_{m+1} and y^*_{n+1} denote the value of x_{m+1} and y_{n+1} in the optimal solution of the respective linear programming problems. It is known from the *strong duality property* given in Sec. 6.1 that $-x^*_{m+1} = -y^*_{n+1}$, so that $x^*_{m+1} = y^*_{n+1}$. However, it is evident from the definition of $\underline{v}$ and $\overline{v}$ that $\underline{v} = x^*_{m+1}$ and $\overline{v} = y^*_{n+1}$, so it follows that $\underline{v} = \overline{v}$, as claimed by the minimax theorem.

One remaining loose end needs to be tied up, namely, what to do about x_{m+1} and y_{n+1} being unrestricted in sign in the linear programming formulations. If it is clear that $v \geq 0$ so that the optimal values of x_{m+1} and y_{n+1} are nonnegative, then it is safe to introduce nonnegativity constraints for these variables for the purpose of applying the simplex method. However, if $v < 0$, then an adjustment needs to be made. One possibility is to use the approach described in Sec. 4.6 for replacing a variable without a nonnegativity constraint by the difference of two nonnegative variables. Another is to reverse players 1 and 2 so that the payoff table would be rewritten as the payoff to the original player 2, which would make the corresponding value of v positive. A third, and the most commonly used, procedure is to add a sufficiently large fixed constant to all the entries in the payoff table that the new value of the game will be positive. (For example, setting this constant equal to the absolute value of the largest negative entry will suffice.) Because this same constant is added to every entry, this adjustment cannot alter the optimal mixed strategies in any way, so they can now be obtained in the usual manner. The indicated value of the game would be increased by the amount of the constant, but this value can be readjusted after the solution has been obtained.

To illustrate this linear programming approach, consider again variation 3 of the political campaign problem after dominated strategy 3 for player 1 is eliminated (see Table 14.6). Because there are some negative entries in the reduced payoff table, it is unclear at the outset whether the *value* of the game v is *nonnegative* (it turns out to be). For the moment, let us assume that $v \geq 0$ and proceed without making any of the adjustments discussed in the preceding paragraph.

To write out the linear programming model for player 1 for this example, note that p_{ij} in the general model is the entry in row i and column j of Table 14.6, for $i = 1, 2$ and $j = 1, 2, 3$. The resulting model is

$$\text{Maximize} \qquad x_3,$$

subject to

$$\begin{aligned} 5x_2 - x_3 &\geq 0 \\ -2x_1 + 4x_2 - x_3 &\geq 0 \end{aligned}$$

$$
\begin{aligned}
2x_1 - 3x_2 - x_3 &\geq 0 \\
x_1 + x_2 &= 1
\end{aligned}
$$

and

$$x_1 \geq 0, \qquad x_2 \geq 0.$$

Applying the simplex method to this linear programming problem (after adding the constraint $x_3 \geq 0$) yields $x_1^* = \frac{7}{11}$, $x_2^* = \frac{4}{11}$, $x_3^* = \frac{2}{11}$ as the optimal solution. (See Probs. 14.5-7 and 14.5-8.) Consequently, just as was found by the graphical procedure in the preceding section, the optimal mixed strategy for player 1 according to the minimax criterion is $(x_1, x_2) = (\frac{7}{11}, \frac{4}{11})$, and the value of the game is $v = x_3^* = \frac{2}{11}$. The simplex method also yields the optimal solution for the dual (given next) of this problem, namely, $y_1^* = 0$, $y_2^* = \frac{5}{11}$, $y_3^* = \frac{6}{11}$, $y_4^* = \frac{2}{11}$, so the optimal mixed strategy for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$.

The dual of the preceding problem is just the linear programming model for player 2 (the one with variables $y_1, y_2, \ldots, y_n, y_{n+1}$) shown earlier in this section. (See Prob. 14.5-6.) By plugging in the values of p_{ij} from Table 14.6, this model is

$$\text{Minimize} \qquad y_4,$$

subject to

$$
\begin{aligned}
- 2y_2 + 2y_3 - y_4 &\leq 0 \\
5y_1 + 4y_2 - 3y_3 - y_4 &\leq 0 \\
y_1 + y_2 + y_3 &= 1
\end{aligned}
$$

and

$$y_1 \geq 0, \qquad y_2 \geq 0, \qquad y_3 \geq 0.$$

Applying the simplex method directly to this model (after adding the constraint $y_4 \geq 0$) yields the optimal solution: $y_1^* = 0$, $y_2^* = \frac{5}{11}$, $y_3^* = \frac{6}{11}$, $y_4^* = \frac{2}{11}$ (as well as the optimal dual solution $x_1^* = \frac{7}{11}$, $x_2^* = \frac{4}{11}$, $x_3^* = \frac{2}{11}$). Thus, the optimal mixed strategy for player 2 is $(y_1, y_2, y_3) = (0, \frac{5}{11}, \frac{6}{11})$, and the value of the game is again seen to be $v = y_4^* = \frac{2}{11}$.

Because we already had found the optimal mixed strategy for player 2 while dealing with the first model, we did not have to solve the second one. In general, you always can find optimal mixed strategies for *both* players by choosing just one of the models (either one) and then using the simplex method to solve for both an optimal solution and an optimal dual solution.

When the simplex method was applied to both of these linear programming models, a nonnegativity constraint was added that assumed that $v \geq 0$. If this assumption were violated, both models would have no feasible solutions, so the simplex method would stop quickly with this message. To avoid this risk, we could have added a positive constant, say, 3 (the absolute value of the largest negative entry), to all the entries in Table 14.6. This then would increase by 3 all the coefficients of x_1, x_2, y_1, y_2, and y_3 in the inequality constraints of the two models. (See Prob. 14.5-1.)

14.6 EXTENSIONS

Although this chapter has considered only two-person, zero-sum games with a finite number of pure strategies, game theory extends far beyond this kind of game. In fact, extensive research has been done on a number of more complicated types of games, including the ones summarized in this section.

The simplest generalization is to the *two-person, constant-sum game.* In this case, the sum of the payoffs to the two players is a fixed constant (positive or negative) regardless of which combination of strategies is selected. The only difference from a two-person, zero-sum game is that, in the latter case, the constant must be zero. A nonzero constant may arise instead because, in addition to one player winning whatever the other one loses, the two players may share some reward (if the constant is positive) or some cost (if the constant is negative) for participating in the game. Adding this fixed constant does nothing to affect which strategies should be chosen. Therefore, the analysis for determining optimal strategies is exactly the same as described in this chapter for two-person, zero-sum games.

A more complicated extension is to the *n-person game,* where more than two players may participate in the game. This generalization is particularly important because, in many kinds of competitive situations, frequently more than two competitors are involved. This may occur, e.g., in competition among business firms, in international diplomacy, and so forth. Unfortunately, the existing theory for such games is less satisfactory than it is for two-person games.

Another generalization is the *nonzero-sum game,* where the sum of the payoffs to the players need not be 0 (or any other fixed constant). This case reflects the fact that many competitive situations include noncompetitive aspects that contribute to the mutual advantage or mutual disadvantage of the players. For example, the advertising strategies of competing companies can affect not only how they will split the market but also the total size of the market for their competing products. However, in contrast to a constant-sum game, the size of the mutual gain (or loss) for the players depends on the combination of strategies chosen.

Because mutual gain is possible, nonzero-sum games are further classified in terms of the degree to which the players are permitted to cooperate. At one extreme is the *noncooperative game,* where there is no preplay communication between the players. At the other extreme is the *cooperative game,* where preplay discussions and binding agreements are permitted. For example, competitive situations involving trade regulations between countries, or collective bargaining between labor and management, might be formulated as cooperative games. When there are more than two players, cooperative games also allow some of or all the players to form coalitions.

Still another extension is to the class of *infinite games,* where the players have an infinite number of pure strategies available to them. These games are designed for the kind of situation where the strategy to be selected can be represented by a *continuous* decision variable. For example, this decision variable might be the time at which to take a certain action, or the proportion of one's resources to allocate to a certain activity, in a competitive situation.

However, the analysis required in these extensions beyond the two-person, zero-sum, finite game is relatively complex and will not be pursued further here.

14.7 CONCLUSIONS

The general problem of how to make decisions in a competitive environment is a very common and important one. The fundamental contribution of game theory is that it provides a basic conceptual framework for formulating and analyzing such problems in simple situations. However, there is a considerable gap between what the theory can handle and the complexity of most competitive situations arising in practice. Therefore, the conceptual tools of game theory usually play just a supplementary role in dealing with these situations.

Because of the importance of the general problem, research is continuing with some success to extend the theory to more complex situations.

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LEARNING AIDS IN YOUR OR COURSEWARE FOR THIS CHAPTER

"Ch. 14—Game Theory" Files for Solving the Examples:

Excel File
LINGO/LINDO File
MPL/CPLEX File

See Appendix 1 for documentation of the software.

PROBLEMS

The symbol to the left of some of the problems (or their parts) has the following meaning.

C: Use the computer with any of the software options available to you (or as instructed by your instructor) to solve the problem.

An asterisk on the problem number indicates that at least a partial answer is given in the back of the book.

14.1-1. The labor union and management of a particular company have been negotiating a new labor contract. However, negotiations have now come to an impasse, with management making a "final" offer of a wage increase of $1.10 per hour and the union making a "final" demand of a $1.60 per hour increase. Therefore, both sides have agreed to let an impartial arbitrator set the wage increase somewhere between $1.10 and $1.60 per hour (inclusively).

The arbitrator has asked each side to submit to her a confidential proposal for a fair and economically reasonable wage increase (rounded to the nearest dime). From past experience, both sides know that this arbitrator normally accepts the proposal of the side that gives the most from its final figure. If neither side changes its final figure, or if they both give in the same amount, then the arbitrator normally compromises halfway between ($1.35 in this case). Each side now needs to determine what wage increase to propose for its own maximum advantage.

Formulate this problem as a two-person, zero-sum game.

14.1-2. Two manufacturers currently are competing for sales in two different but equally profitable product lines. In both cases the sales volume for manufacturer 2 is three times as large as that for manufacturer 1. Because of a recent technological breakthrough, both manufacturers will be making a major improvement in both products. However, they are uncertain as to what development and marketing strategy to follow.

If both product improvements are developed simultaneously, either manufacturer can have them ready for sale in 12 months. Another alternative is to have a "crash program" to develop only one product first to try to get it marketed ahead of the competition. By doing this, manufacturer 2 could have one product ready for sale in 9 months, whereas manufacturer 1 would require 10 months (because of previous commitments for its production facilities). For either manufacturer, the second product could then be ready for sale in an additional 9 months.

For either product line, if both manufacturers market their improved models simultaneously, it is estimated that manufacturer 1 would increase its share of the total future sales of this product by 8 percent of the total (from 25 to 33 percent). Similarly, manufacturer 1 would increase its share by 20, 30, and 40 percent of the total if it marketed the product sooner than manufacturer 2 by 2, 6, and 8 months, respectively. On the other hand, manufacturer 1 would lose 4, 10, 12, and 14 percent of the total if manufacturer 2 marketed it sooner by 1, 3, 7, and 10 months, respectively.

Formulate this problem as a two-person, zero-sum game, and then determine which strategy the respective manufacturers should use according to the minimax criterion.

14.1-3. Consider the following parlor game to be played between two players. Each player begins with three chips: one red, one white, and one blue. Each chip can be used only once.

To begin, each player selects one of her chips and places it on the table, concealed. Both players then uncover the chips and determine the payoff to the winning player. In particular, if both players play the same kind of chip, it is a draw; otherwise, the following table indicates the winner and how much she receives from the other player. Next, each player selects one of her two remaining chips and repeats the procedure, resulting in another payoff according to the following table. Finally, each player plays her one remaining chip, resulting in the third and final payoff.

Winning Chip	Payoff ($)
Red beats white	50
White beats blue	40
Blue beats red	30
Matching colors	0

Formulate this problem as a two-person, zero-sum game by identifying the form of the strategies and payoffs.

14.2-1. Reconsider Prob. 14.1-1.

(a) Use the concept of dominated strategies to determine the best strategy for each side.

(b) Without eliminating dominated strategies, use the minimax criterion to determine the best strategy for each side.

14.2-2.* For each of the following payoff tables, determine the optimal strategy for each player by successively eliminating dominated strategies. (Indicate the order in which you eliminated strategies.)

(a)

		Player 2		
Strategy		**1**	**2**	**3**
Player 1	1	−3	1	2
	2	1	2	1
	3	1	0	−2

(b)

		Player 2		
Strategy		**1**	**2**	**3**
Player 1	1	1	2	0
	2	2	−3	−2
	3	0	3	−1

14.2-3. Consider the game having the following payoff table.

		Player 2			
Strategy		**1**	**2**	**3**	**4**
Player 1	1	2	−3	−1	1
	2	−1	1	−2	2
	3	−1	2	−1	3

Determine the optimal strategy for each player by successively eliminating dominated strategies. Give a list of the dominated strategies (and the corresponding dominating strategies) in the order in which you were able to eliminate them.

14.2-4. Find the saddle point for the game having the following payoff table.

		Player 2		
Strategy		**1**	**2**	**3**
Player 1	1	1	−1	1
	2	−2	0	3
	3	3	1	2

Use the minimax criterion to find the best strategy for each player. Does this game have a saddle point? Is it a stable game?

14.2-5. Find the saddle point for the game having the following payoff table.

	Strategy	Player 2			
		1	2	3	4
	1	3	−3	−2	−4
Player 1	2	−4	−2	−1	1
	3	1	−1	2	0

Use the minimax criterion to find the best strategy for each player. Does this game have a saddle point? Is it a stable game?

14.2-6. Two companies share the bulk of the market for a particular kind of product. Each is now planning its new marketing plans for the next year in an attempt to wrest some sales away from the other company. (The total sales for the product are relatively fixed, so one company can increase its sales only by winning them away from the other.) Each company is considering three possibilities: (1) better packaging of the product, (2) increased advertising, and (3) a slight reduction in price. The costs of the three alternatives are quite comparable and sufficiently large that each company will select just one. The estimated effect of each combination of alternatives on the *increased percentage of the sales* for company 1 is as follows:

	Strategy	Player 2		
		1	2	3
	1	2	3	1
Player 1	2	1	4	0
	3	3	−2	−1

Each company must make its selection before learning the decision of the other company.

(a) Without eliminating dominated strategies, use the minimax (or maximin) criterion to determine the best strategy for each company.

(b) Now identify and eliminate dominated strategies as far as possible. Make a list of the dominated strategies, showing the order in which you were able to eliminate them. Then show the resulting reduced payoff table with no remaining dominated strategies.

14.2-7.* Two politicians soon will be starting their campaigns against each other for a certain political office. Each must now select the main issue she will emphasize as the theme of her campaign. Each has three advantageous issues from which to choose, but the relative effectiveness of each one would depend upon the issue chosen by the opponent. In particular, the estimated increase in the vote for politician 1 (expressed as a percentage of the total vote) resulting from each combination of issues is as follows:

		Issue for Politician 2		
		1	2	3
	1	7	−1	3
Issue for Politician 1	2	1	0	2
	3	−5	−3	−1

However, because considerable staff work is required to research and formulate the issue chosen, each politician must make her own choice before learning the opponent's choice. Which issue should she choose?

For each of the situations described here, formulate this problem as a two-person, zero-sum game, and then determine which issue should be chosen by each politician according to the specified criterion.

(a) The current preferences of the voters are very uncertain, so each additional percent of votes won by one of the politicians has the same value to her. Use the minimax criterion.

(b) A reliable poll has found that the percentage of the voters currently preferring politician 1 (before the issues have been raised) lies between 45 and 50 percent. (Assume a uniform distribution over this range.) Use the concept of dominated strategies, beginning with the strategies for politician 1.

(c) Suppose that the percentage described in part (*b*) actually were 45 percent. Should politician 1 use the minimax criterion? Explain. Which issue would you recommend? Why?

14.2-8. Briefly describe what you feel are the advantages and disadvantages of the minimax criterion.

14.3-1. Consider the following parlor game between two players. It begins when a referee flips a coin, notes whether it comes up heads or tails, and then shows this result to player 1 only. Player 1 may then (1) pass and thereby pay $5 to player 2 or (2) bet. If player 1 passes, the game is terminated. However, if he bets, the game continues, in which case player 2 may then either (1) pass and thereby pay $5 to player 1 or (2) call. If player 2 calls, the referee then shows him the coin; if it came up heads, player 2 pays $10 to player 1; if it came up tails, player 2 receives $10 from player 1.

(a) Give the pure strategies for each player. (*Hint:* Player 1 will have four pure strategies, each one specifying how he would respond to each of the two results the referee can show him; player 2 will have two pure strategies, each one specifying how he will respond if player 1 bets.)

(b) Develop the payoff table for this game, using expected values for the entries when necessary. Then identify and eliminate any dominated strategies.

(c) Show that none of the entries in the resulting payoff table are a saddle point. Then explain why any fixed choice of a pure strategy for each of the two players must be an unstable solution, so mixed strategies should be used instead.

(d) Write an expression for the expected payoff in terms of the probabilities of the two players using their respective pure strategies. Then show what this expression reduces to for the following three cases: (i) Player 2 definitely uses his first strategy, (ii) player 2 definitely uses his second strategy, (iii) player 2 assigns equal probabilities to using his two strategies.

14.4-1. Reconsider Prob. 14.3-1. Use the graphical procedure described in Sec. 14.4 to determine the optimal mixed strategy for each player according to the minimax criterion. Also give the corresponding value of the game.

14.4-2. Consider the game having the following payoff table.

		Player 2	
Strategy		**1**	**2**
Player 1	1	3	−2
	2	−1	2

Use the graphical procedure described in Sec. 14.4 to determine the value of the game and the optimal mixed strategy for each player according to the minimax criterion. Check your answer for player 2 by constructing *his* payoff table and applying the graphical procedure directly to this table.

14.4-3.* For each of the following payoff tables, use the graphical procedure described in Sec. 14.4 to determine the value of the game and the optimal mixed strategy for each player according to the minimax criterion.

(a)

		Player 2		
Strategy		**1**	**2**	**3**
Player 1	1	4	3	1
	2	0	1	2

(b)

		Player 2		
Strategy		**1**	**2**	**3**
Player 1	1	1	−1	3
	2	0	4	1
	3	3	−2	5
	4	−3	6	−2

14.4-4. The A. J. Swim Team soon will have an important swim meet with the G. N. Swim Team. Each team has a star swimmer (John and Mark, respectively) who can swim very well in the 100-yard butterfly, backstroke, and breaststroke events. However, the rules prevent them from being used in more than two of these events. Therefore, their coaches now need to decide how to use them to maximum advantage.

Each team will enter three swimmers per event (the maximum allowed). For each event, the following table gives the best time previously achieved by John and Mark as well as the best time for each of the other swimmers who will definitely enter that event. (Whichever event John or Mark does not swim, his team's third entry for that event will be slower than the two shown in the table.)

	A. J. Swim Team			G. N. Swim Team		
	Entry			Entry		
	1	**2**	**John**	**Mark**	**1**	**2**
Butterfly stroke	1:01.6	59.1	57.5	58.4	1:03.2	59.8
Backstroke	1:06.8	1:05.6	1:03.3	1:02.6	1:04.9	1:04.1
Breaststroke	1:13.9	1:12.5	1:04.7	1:06.1	1:15.3	1:11.8

The points awarded are 5 points for first place, 3 points for second place, 1 point for third place, and none for lower places. Both coaches believe that all swimmers will essentially equal their best times in this meet. Thus, John and Mark each will definitely be entered in two of these three events.

(a) The coaches must submit all their entries before the meet without knowing the entries for the other team, and no changes are permitted later. The outcome of the meet is very uncertain, so each additional point has equal value for the coaches. Formulate this problem as a two-person, zero-sum game. Eliminate dominated strategies, and then use the graphical procedure described in Sec. 14.4 to find the optimal mixed strategy for each team according to the minimax criterion.

(b) The situation and assignment are the same as in part (*a*), except that both coaches now believe that the A. J. team will win the swim meet if it can win 13 or more points in these three events, but will lose with less than 13 points. [Compare the resulting optimal mixed strategies with those obtained in part (*a*).]

(c) Now suppose that the coaches submit their entries during the meet one event at a time. When submitting his entries for an event, the coach does not know who will be swimming that event for the other team, but he does know who has swum in *preceding* events. The three key events just discussed are swum in the order listed in the table. Once again, the A. J. team needs 13 points in these events to win the swim meet. Formulate this problem as a two-person, zero-sum game. Then use the concept of dominated strategies to determine the best strategy for the G. N. team that actually "guarantees" it will win under the assumptions being made.

(d) The situation is the same as in part (*c*). However, now assume that the coach for the G. N. team does not know about game theory and so may, in fact, choose any of his available strategies that have Mark swimming two events. Use the concept of dominated strategies to determine the best strategies from which the coach for the A. J. team should choose. If this coach knows that the other coach has a tendency to enter Mark in the butterfly and the backstroke more often than in the breaststroke, which strategy should she choose?

14.5-1. Refer to the last paragraph of Sec. 14.5. Suppose that 3 were added to all the entries of Table 14.6 to ensure that the corresponding linear programming models for both players have feasible solutions with $x_3 \geq 0$ and $y_4 \geq 0$. Write out these two models. Based on the information given in Sec. 14.5, what are the optimal solutions for these two models? What is the relationship between x_3^* and y_4^*? What is the relationship between the *value* of the original game v and the values of x_3^* and y_4^*?

14.5-2.* Consider the game having the following payoff table.

Strategy		Player 2: 1	2	3	4
Player 1	1	5	0	3	1
	2	2	4	3	2
	3	3	2	0	4

(a) Use the approach described in Sec. 14.5 to formulate the problem of finding optimal mixed strategies according to the minimax criterion as a linear programming problem.

C **(b)** Use the simplex method to find these optimal mixed strategies.

14.5-3. Follow the instructions of Prob. 14.5-2 for the game having the following payoff table.

Strategy		Player 2: 1	2	3
Player 1	1	4	2	−3
	2	−1	0	3
	3	2	3	−2

14.5-4. Follow the instructions of Prob. 14.5-2 for the game having the following payoff table.

Strategy		Player 2: 1	2	3	4	5
Player 1	1	1	−3	2	−2	1
	2	2	3	0	3	−2
	3	0	4	−1	−3	2
	4	−4	0	−2	2	−1

14.5-5. Section 14.5 presents a general linear programming formulation for finding an optimal mixed strategy for player 1 and for player 2. Using Table 6.14, show that the linear programming problem given for player 2 is the dual of the problem given for player 1. (*Hint:* Remember that a dual variable with a nonpositivity constraint $y_i' \leq 0$ can be replaced by $y_i = -y_i'$ with a nonnegativity constraint $y_i \geq 0$.)

14.5-6. Consider the linear programming models for players 1 and 2 given near the end of Sec. 14.5 for variation 3 of the political campaign problem (see Table 14.6). Follow the instructions of Prob. 14.5-5 for these two models.

14.5-7. Consider variation 3 of the political campaign problem (see Table 14.6). Refer to the resulting linear programming model for player 1 given near the end of Sec. 14.5. Ignoring the objective function variable x_3, plot the *feasible region* for x_1 and x_2 graphically (as described in Sec. 3.1). (*Hint:* This feasible region consists of a single line segment.) Next, write an algebraic expression for the maximizing value of x_3 for any point in this feasible region. Finally, use this expression to demonstrate that the optimal solution must, in fact, be the one given in Sec. 14.5.

C **14.5-8.** Consider the linear programming model for player 1 given near the end of Sec. 14.5 for variation 3 of the political campaign problem (see Table 14.6). Verify the optimal mixed strategies for both players given in Sec. 14.5 by applying an automatic routine for the simplex method to this model to find both its optimal solution and its optimal dual solution.

14.5-9. Consider the general $m \times n$, two-person, zero-sum game. Let p_{ij} denote the payoff to player 1 if he plays his strategy i ($i = 1, \ldots, m$) and player 2 plays her strategy j ($j = 1, \ldots, n$). Strategy 1 (say) for player 1 is said to be *weakly dominated* by strategy 2 (say) if $p_{1j} \leq p_{2j}$ for $j = 1, \ldots, n$ and $p_{1j} = p_{2j}$ for one or more values of j.

(a) Assume that the payoff table possesses one or more saddle points, so that the players have corresponding optimal pure strategies under the minimax criterion. Prove that eliminating *weakly dominated* strategies from the payoff table cannot eliminate all these saddle points and cannot produce any new ones.

(b) Assume that the payoff table does not possess any saddle points, so that the optimal strategies under the minimax criterion are mixed strategies. Prove that eliminating weakly dominated pure strategies from the payoff table cannot eliminate all optimal mixed strategies and cannot produce any new ones.